Toward Higher Algebra Over $\mathbb{F}_1$

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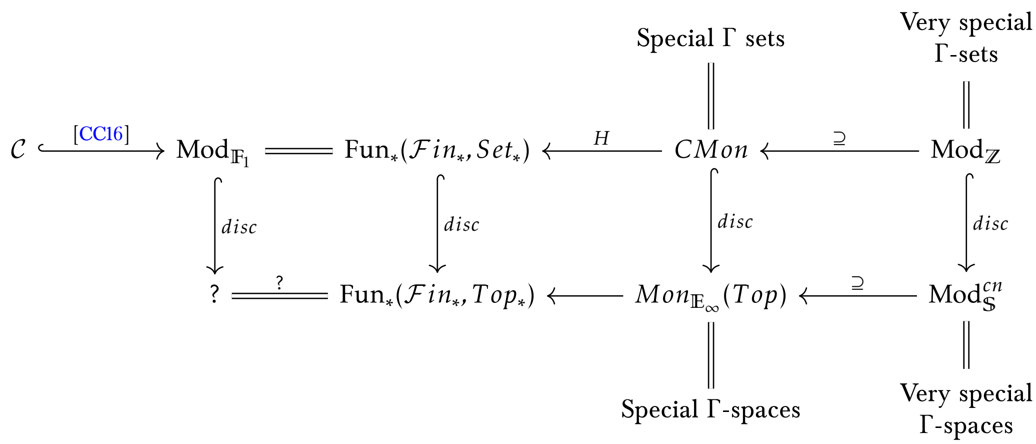
This talk represents joint work with Joe Moeller.

1 Big Picture

During this talk, I'll try to explain the diagram below. Much of this is speculative, but at the end I'll describe a result which a first step toward understanding what the bottom left corner should be. Much of what I'll talk about here is described in [CC16, CC20] but of course is older than that, cf. [DGM13, Seg74].



First of all, let me mention some of the things that, as a result of [CC16], we can put in place of $\mathcal{C}$, many of which are existing models for "algebra over $\mathbb{F}_1$," and some of which show up in number theory and elsewhere. From the right hand side of the diagram we already see that $\mathcal{C}$ can be either commutative monoids or Abelian groups (via the functor H, which I'll describe later), but there are more exotic structures in there as well. Some of the things $\mathcal{C}$ can be are:

1. Quotient hypergroups, i.e. hypergroups that are obtained by taking a commutative ring R, a subgroup $G < GL_1(R)$, and forming the quotient R/G (as a set). In fact this construction always yields hyper*rings*, since quotienting destroys the additive structure but not the multiplicative structure. This includes the adèle class hyperring $\mathbb{A}_k/k^*$, for a global field k, of Connes, Consani and Marcolli [CCM09]. I'll say more about this in a bit.

2. Durov's generalized rings [Dur07] (which include a model of $\mathbb{F}_1$ which is *not* the one Connes and Consani use).

3. Partial groups, i.e. sets with partially defined Abelian group structures.

4. Arakelov divisors of $\overline{Spec(\mathbb{Z})}$, although Connes and Consani give these as *sheaves* of $\mathbb{F}_1$-modules, so there's something a bit more general going on there that has to be taken into account.

5. There's also a "monoid ring" construction which takes a (not necessarily commutative) monoid M and produces an $\mathbb{F}_1$-algebra $\mathbb{F}_1[M]$.

2 Specialness and the Functor H

Definition 2.1. Write $\Gamma\mathrm{Set}_*$ for the category $\mathrm{Fun}_*(\mathcal{F}in_*, Set_*)$ and $\Gamma\mathrm{Top}_*$ for the category $\mathrm{Fun}_*(\mathcal{F}in_*, \mathrm{Top}_*)$. In general, if $\mathcal{C}$ is a pointed category then I'll write $\Gamma\mathcal{C}$ for the functor category $\mathrm{Fun}_*(\mathcal{F}in_*, \mathcal{C})$. For instance, $\Gamma\mathrm{sSet}_*$, the category of pointed functors from $\mathcal{F}in_*$ to pointed simplicial sets, will come up later.

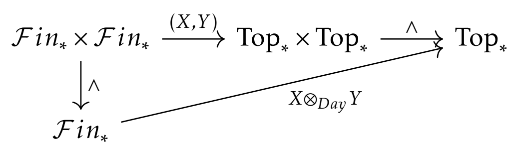
Remark 2.2. The reason for the symbol Γ showing up here is that in [Seg74], where these ideas first appeared, a somewhat complicated category Γ was defined and then functors $\Gamma^{op} \to \mathrm{Top}_*$ were considered. Later on it was determined that in fact $\Gamma^{op} \simeq \mathcal{F}in_*$, but the terminology stuck.

Definition 2.3. Write $\langle n \rangle = \{*, 1, 2, \dots, n\}$ for the objects of $\mathcal{F}in_*$. Then write:

1. $\rho_i \colon \langle n \rangle \to \langle 1 \rangle$ for the map given by
$$\rho_i(j) = \begin{cases} 1 & i = j \\ * & i \neq j \end{cases}$$
2. $\mu \colon \langle 2 \rangle \to \langle 1 \rangle$ for the map given by $\mu(1) = \mu(2) = 1$.

Proposition 2.4. *The category $\mathcal{F}in_*$ has a symmetric monoidal structure given by the smash product $\langle n \rangle \wedge \langle m \rangle = \langle nm \rangle$. This induces a Day convolution monoidal structure on $\Gamma\mathrm{Top}_*$.*

Remark 2.5. Recall that the Day convolution structure on $\Gamma\mathrm{Top}_*$ is given by defining the Day tensor product of X and Y to be the left Kan extension of $X \wedge Y$ (the pointwise smash product taken in Top) along the tensor product of $\mathcal{F}in_*$ itself:



Remark 2.6. Note that for pointed category $\mathcal{C}$ with finite pullbacks and any $X \in \Gamma\mathcal{C}$, the maps $X(\rho_i) \colon X\langle n \rangle \to X\langle 1 \rangle$ assemble into a map $T_n \colon X\langle n \rangle \to \prod_n X\langle 1 \rangle$. We'll call this map the n^{th} Segal map.

Definition 2.7. We say that $X \in \Gamma\mathrm{Top}_*$ is *special* if each T_n is a weak equivalence.

Remark 2.8. Note of course that $X \in \Gamma\mathrm{Set}_*$ being special simply means that the maps T_n are all isomorphisms.

Proposition 2.9. *If $X \in \Gamma\mathrm{Top}_*$ is special then $\pi_0 X\langle 1 \rangle$ is canonically equipped with a commutative monoid structure.*

Proof. The unit map is given by applying X to the unique map $\langle 0 \rangle \to \langle 1 \rangle$ and using that X is pointed. The multiplication map is the composite
$$\pi_0 X\langle 1 \rangle \times \pi_0 X\langle 1 \rangle \cong \pi_0 X\langle 2 \rangle \xrightarrow{X(\mu)} \pi_0 X\langle 1 \rangle.$$
The associativity and commutativity diagrams all follow from functoriality. □

Remark 2.10. A "fun" exercise is to sit down and prove that, in fact, a special Γ-space is a homotopy commutative monoid in Top. Even better, though this is much harder to prove, it turns out that special Γ-spaces are exactly the same thing as $\mathbb{E}_\infty$-monoids in Top. But for what we're doing here we'll only need the structure on π_0.

Definition 2.11. If $X \in \Gamma\mathrm{Top}_*$ is special, then we say that X is *very special* if $\pi_0 X\langle 1 \rangle$ is a group with respect to the monoid structure described above.

Remark 2.12. Another word for very special Γ-spaces is *grouplike $\mathbb{E}_\infty$-spaces*, i.e. infinite loop spaces, i.e. connective spectra. This is what's going on in the bottom right corner of the diagram we started with. For the top right part, note that a special Γ-set is exactly a commutative monoid, and a very special Γ-set is the data of an Abelian group.

Remark 2.13. Another "fun" exercise is to check that every map in $\mathcal{F}in_*$ can be constructed by taking coproducts of (I don't think you even need this much data):

1. identity maps;
2. the zero map $*: \langle 1\rangle \to \langle 0\rangle$;
3. the inclusion $i_n: \langle 1\rangle \to \langle n\rangle$ and $i_0: \langle 0\rangle \to \langle 1\rangle$;
4. the twist map $\tau: \langle 2\rangle \to \langle 2\rangle$;
5. the "multiplication" map $\mu: \langle 2\rangle \to \langle 1\rangle$;
6. and the projection $\rho_1: \langle 2\rangle \to \langle 1\rangle$.

Definition 2.14. Write $H: CMon \to \Gamma\mathrm{Top}_*$ for the functor defined by the following data:

1. $HM\langle n\rangle = M^n$ (with $M^0 = *$).
2. $HM(i_n)$ is the inclusion into the first coordinate and $HM(i_0)$ is the unit map of M.
3. Permutations permute the coordinates of M.
4. $HM(\mu)$ is the multiplication map of M.
5. ρ_1 projects onto the first coordinate.

Proposition 2.15. *Some facts about H are:*

1. *H factors through special Γ-sets.*
2. *If M is an Abelian group then HM is very special.*
3. *H is faithful and lax monoidal with respect to the product in $CMon$ and the Day convolution monoidal structure in $\Gamma\mathrm{Top}_*$.*

Remark 2.16. I'm following [CC16] here in calling this functor H despite the fact that H is typically reserved for the Eilenberg-MacLane spectrum functor. This might be confusing until you realize that under the equivalence between very special Γ-spaces and connective spectra, HM really does go to the Eilenberg-MacLane spectrum of M when M is an Abelian group.

Remark 2.17. The nice thing about H being lax monoidal is that it takes algebras to algebras. So in particular rings and semi-rings give nice examples of monoids in the category of Γ-sets, which Connes and Consani call $\mathbb{F}_1$-algebras.

So this tells you how classical algebra "lives inside of" algebra over $\mathbb{F}_1$. The functor HM should be compared to the usual bar construction of a monoid M. Recall that the bar construction BM is a simplicial set with $BM_n = M^n$, the outer face maps are given by projection onto the internal coordinates, the inner face maps are given by applying the multiplication map to two internal coordinates, and the degeneracies are given by using the unit map of M. Alternatively, the bar construction is the nerve of the category with one object whose set of morphisms is M.

More generally, given any $\mathbb{E}_n$-monoid M (in a nice enough category), we can produce the bar construction BM as a simplicial object. In general, the colimit of the bar construction of an $\mathbb{E}_n$-monoid is an $\mathbb{E}_{n-1}$-monoid. But if M is an $\mathbb{E}_\infty$-monoid (e.g. a strictly commutative monoid) then we can infinitely iterate this construction. This will come up later.

3 Quotient Hyperrings as $\mathbb{F}_1$-algebras

As an example of some more "exotic" structure in this framework, I'm now going to tell how to encode certain hyperstructures as $\mathbb{F}_1$-modules. First we'll need some definitions.

Definition 3.1. A hypergroup is a set G with a multivalued binary operations $\boxplus\colon G\times G\to\mathcal{P}(G)-\varnothing$ satisfying the following properties:

1. There is a distinguished element $0\in G$ such that $0\boxplus g=\{g\}=g\boxplus 0$ for all $g\in G$.
2. If $(g\boxplus h)\boxplus k=\cup_{x\in g\boxplus h}x\boxplus k$, and similarly for the other parenthesization, then $\boxplus$ is associative.
3. For every $g\in G$ there exists a unique h such that $0\in g\boxplus h$, and we write $-g$ for this element.
4. For all $g,h,k\in G$, $g\in h\boxplus k$ implies that $k\in g\boxplus(-h)$.

If $g\boxplus h=h\boxplus g$ then we call G an Abelian hypergroup.

Proposition 3.2. *There is a category of Abelian hypergroups with a symmetric monoidal structure that restricts to usual tensor product of $\mathbb{Z}$-modules when restricted to the subcategory of Abelian groups.*

Definition 3.3. A *hyperring* is a monoid object in the category of Abelian hypergroups.

Example 3.4 (Quotient Hyperrings). Let R be a commutative ring and G a subgroup of $GL_1(R)$. Then the set quotient R/G has a hypergroup structure given by $[x]\boxplus[y]=\{[x+gy]\in R/G\colon g\in G\}$. You can check that the multiplication on R descends unharmed to the quotient R/G, making R/G into a hyperring.

Some more specific examples are:

1. The quotient $\mathbb{Z}/C_2$ by the sign action. The set of objects is $\mathbb{N}$, but the hyperaddition is given by $[n]\boxplus[m]=\{[n+m],[n-m]\}$. Note that we have to take the *equivalence classes* of the sum and difference to make sure that we only have elements of $\mathbb{N}$ in our subset.
2. The quotient $(\mathbb{Z}/3)/C_2$ by the sign action. The underlying set is $\mathbb{Z}/2$, but the hyperaddition is $[1]+[1]=\{[0],[1]\}$. This is the so-called *Krasner hyperring*. In [Kra57], Marc Krasner used hyperrings in an approach to local class field theory.
3. Connes, Consani and Marcolli's adèle class space $\mathbb{A}_k/k^*$ described earlier.

Remark 3.5. Note that if $G<GL_1(R)$ then G also acts on R^n via the diagonal:

$$G\times R^n\overset{\Delta^n\times R^n}{\to}G^n\times R^n\to R^n$$

Furthermore, because $G<GL_1(R)$ you can check that this extends to an action of G on the entire functor HR, i.e. there is a functor $BG\to\Gamma\mathrm{Set}_*$ taking the base point of BG to HR. Checking this comes down to just making sure that all of the structure maps of HR are G-equivariant.

Proposition 3.6. *The hypergroup structure of R/G can be recovered from the the orbits functor HR/G.*

Proof. The two maps $\rho_1,\rho_2\colon(R^2)/G\to R/G$ induce the Segal map $T_2\colon(R^2)/G\to R/G\times R/G$. Given two classes $[x],[y]\in R/G$, write $T_2^{-1}([x],[y])$ for their preimage (which is a subset of $(R^2)/G$) under the Segal map. Then define $[x]\boxplus[y]=\mu(T_2^{-1}([x],[y]))$, which again is now a subset of R/G. It's not too hard to see that this is precisely the hypergroup structure defined previously. □

Remark 3.7. Note that the hypergroup structure described above is structurally the same as the group structure on HR, in the sense that both are given by applying μ to the inverse image of the Segal map. It's just that in the case of HR, or any very special Γ-set, T_2 is a bijection.

4 Looping and Delooping

The equivalence between very special Γ-spaces and connective spectra is given by the "infinite delooping" functor:

$$\{\text{v.s. }\Gamma\text{-spaces}\} \xrightarrow{B^\infty} \mathrm{Mod}_{\mathbb{S}}^{cn}$$

which takes a very special Γ-space X to the Ω-spectrum $\{X, BX, B^2X, \ldots\}$. Here we're using two facts:

1. Being $\mathbb{E}_\infty$-monoids in Top, we can take the bar construction of very special Γ-spaces (i.e. "deloop" them).
2. The bar construction on a very special Γ-space is again a very special Γ-space, so we can iterate this construction.

Note that when we restrict B^∞ to very special Γ-*sets*, i.e. Abelian groups, we get the functor which takes an Abelian group to its Eilenberg-MacLane spectrum.

Remark 4.1. One can also form the bar construction of a (not necessarily very) special Γ-space, and get another special Γ-space (in fact a very special one). The problem here is that this does not qualify as a "delooping" because if we take ΩX (where by Ω on the functor X I mean applying the loop space functor Ω pointwise, i.e. $\Omega X = \Omega \circ X$) then we don't recover X, we recover its "group completion." So B^∞ only becomes an equivalence of (∞-)categories when restricted to very special Γ-spaces.

Let me say something more about the above remark. If we'd like to construct something like an Eilenberg-MacLane spectrum of an $\mathbb{F}_1$-module, then we'll want to be able to infinitely deloop them. In particular, we'd like to be able to deloop them at least once, i.e. take an $\mathbb{F}_1$-module A and produce a Γ-space BA such that $\Omega BA \cong A$ for some reasonable notion of Ω.

Let's think for a moment about some concrete examples. Abelian groups are one kind of $\mathbb{F}_1$-module. Given a discrete Abelian group G, we can construct BG, as a simplicial set, by taking the nerve of the category with one object $*$ and morphisms $BG(*,*) = G$, with the composition operation given by the group law of G. This is a simplicial set with one 0-simplex, generators of G for 1-simplices, 2-simplices enforcing the relations between the generators of G, and only degenerate simplices in higher degrees. Then, "loops" on this object is precisely G (regarded as a constant simplicial group). Equivalently, we could think of BG as a category, and loops on it would correspond to taking the set of endomorphisms of $*$, hence G. In this case, thinking of BG as a space and thinking of BG as a simplicial set give you the same information. Taking geometric realization of the simplicial set BG replaces all the 1-simplices with paths (which are "invertible"), but they were already invertible in BG. In other words, $\Omega|BG| \cong G$ and $\Omega BG \cong G$ for suitable notions of Ω, and I'm thinking of G either as the space with the discrete topology or as the constant simplicial set.

Now consider a commutative monoid M. We can do the same thing here, take the category BM with one object $*$ and morphisms $BM = M$ with composition given the product in M. This gives a simplicial set by taking the nerve. If we think about this simplicial set as a category (indeed, it's a quasicategory) then we recover M by taking some suitable notion of endomorphisms of the object. On the other hand, if we take geometric realization to get a space $|BM|$ then we're in trouble, because the space $|BM|$ doesn't remember the directions of the morphisms in BM. So when we take Ω of this space, we'll get M back but with every element formally inverted, i.e. the group completion of M. This suggests that, at the very least, if our theory of delooping $\mathbb{F}_1$-modules is going to work, we'll need to replace $\Gamma\mathrm{Top}_*$ with $\Gamma\mathrm{sSet}_*$.

Definition 4.2. Let $\Gamma\mathrm{sSet}_*$ denote the category of pointed functors $\mathrm{Fun}_*(\mathcal{F}in_*, \mathrm{sSet}_*)$. Note that this category is equivalent to $s\mathrm{Mod}_{\mathbb{F}_1}$, the category of simplicial $\mathbb{F}_1$-modules.

This is not a particularly egregious revision. Many people who want to work with connective spectra, or infinite loop spaces, in the style of Segal, have already replaced Top with sSet_*. Luckily, the delooping functor is defined generally on both $\Gamma\mathrm{Top}_*$ and $\Gamma\mathrm{sSet}_*$, and doesn't require any kind of specialness of its input. To keep things simple here, I'll only define it as a functor $\mathrm{Mod}_{\mathbb{F}_1} \to \Gamma\mathrm{sSet}_*$, rather than on all of $\Gamma\mathrm{sSet}_*$.

Definition 4.3. Let $\chi\colon \Delta^{op} \to \mathcal{F}in_*$ denote the functor determined by noticing that the pointed simplicial set $\Delta^1/\partial\Delta^1\colon \Delta^{op} \to Set_*$ is finite in every degree.

Remark 4.4. The simplicial set $\Delta^1/\partial\Delta^1$ is a simplicial model for S^1, but it is not a Kan complex, so I'll avoid writing S^1 for it since, in our situation, we need to be quite careful about keeping spaces and simplicial sets separate.

Remark 4.5. A "fun" exercise is to work out that the simplicial set $HM \circ \chi$ is precisely the bar construction of M.

Definition 4.6. For $X \in \mathrm{Mod}_{\mathbb{F}_1}$, define BX by the composite functor

$$\mathrm{Mod}_{\mathbb{F}_1} \xrightarrow{\wedge^*} \mathrm{Fun}_*(\mathcal{F}in_*, \mathrm{Mod}_{\mathbb{F}_1}) = \Gamma\,\mathrm{Mod}_{\mathbb{F}_1} \xrightarrow{\chi^*} \mathrm{Fun}(\Delta^{op}, \mathrm{Mod}_{\mathbb{F}_1}) \simeq \Gamma\mathrm{sSet}_*$$

where $\wedge^*$ denotes the functor which takes a Γ-set X to the functor which takes $\langle n\rangle$ to $(\langle m\rangle \mapsto X\langle nm\rangle)$.

Remark 4.7. You can probably see that if you were to define this functor on all of $\Gamma\mathrm{sSet}_*$ instead of just $\mathrm{Mod}_{\mathbb{F}_1}$, you'd land in $\Gamma\mathrm{ss}Set_*$. To end up back in $\Gamma\mathrm{sSet}_*$. you'd have to take geometric realization (not in the sense that you'd land in Top_*, though you could do that too, but in the sense of the generalized geometric realization for simplicial objects in any nice category). Luckily for bisimplicial sets this is the same as taking the diagonal simplicial set. When your bisimplicial set is discrete along one axis, this is the same as doing nothing at all.

Remark 4.8. Any object $X \in \Gamma\mathrm{sSet}_*$ has an "underlying" simplicial set $X\langle 1\rangle$. In the case of $X = HM$, $BHM\langle 1\rangle$ is the simplicial set BM. In this case you can recover M just from the data of that simplicial set, but this isn't generally true.

Proposition 4.9. *Let HR/G be the $\mathbb{F}_1$-module associated to a quotient hypergroup R/G. The G-action on HR lifts to a G-action on BHR and $B(HR/G) \cong (BHR)/G$.*

Remark 4.10. One intuitive reason for the above is that both the bar construction and taking G-orbits are colimits so they should suitably commute.

Now that we've got "delooping," we need "looping." The following definition is used in [CC20] but originally comes from some notes by John Moore called "Algebraic Homotopy Theory."

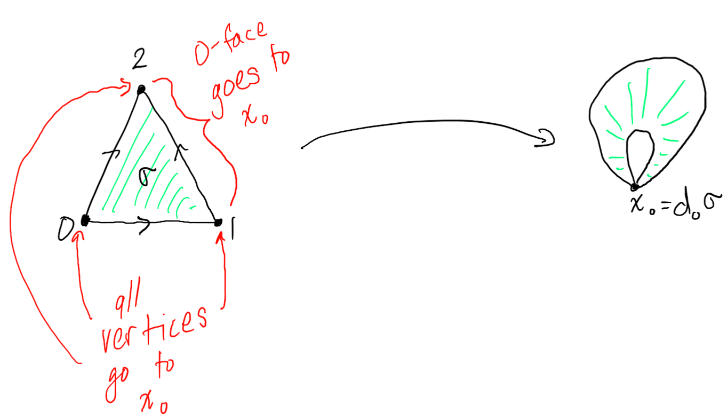
Definition 4.11. Let $K \in \mathrm{sSet}_*$ with basepoint 0-simplex x_0. Define $\Omega_L K$ to be the simplicial set with

$$(\Omega_L K)_n = \{\sigma \in K_{n+1} : d_0\sigma = s_0^n x_0,\ d_{i_0} d_{i_1} \dots d_{i_n}\sigma = x_0\}$$

with face and degeneracy maps the same as K, but shifted.

Remark 4.12. What the condition $d_{i_0} \dots d_{i_n}\sigma = x_0$ means in the above definition is that every vertex of the simplex σ is the basepoint. So what the above definition is essentially saying is: the n-simplices of $\Omega_L K$ are the $n+1$-simplices of K whose 0-face (i.e. the face opposite the 0-vertex) is the degenerate image of the basepoint, and all of whose vertices are the basepoint.

A 0-simplex of $\Omega_L K$ really is just going to be a 1-simplex of K which begins and ends at the basepoint. A 1-simplex is a path between paths and looks sort of like this:



Note that this is *biased* in the sense that we could have asked for the 2-face to be degenerate (instead of the 0-face). If you apply this construction to a quasicategory (hence, in particular, a Kan complex) it corresponds to producing a fibration of simplicial sets $PK \to K$ whose fiber over $x \in K_0$ is the simplicial set of paths in K that end at x (i.e. the slice quasicategory over x) and then taking the fiber of this fibration along the inclusion $x_0 \colon \Delta^0 \to K$. In Lurie's *Higher Topos Theory* this exact same construction is written as $Hom^L(x_0, x_0)$. So this is something like "the space of endomorphisms of x_0." Lurie also constructs something called $Hom^R(x_0, x_0)$ and $Hom(x_0, x_0)$. In the case that K is a quasicategory these are all Kan complexes and homotopy equivalent, but we're not necessarily working with quasicategories here.

Definition 4.13. Define the loop space of $X \in \Gamma\mathrm{sSet}_*$, denoted $\Omega_L X$, to be the composite $\Omega_L \circ X$.

This then lets me state the main theorem.

Theorem 4.14 (Beardsley-Moeller). *Let $X \in \mathrm{Mod}_{\mathbb{F}_1}$ be either: $\mathbb{F}_1$ itself; HM for M some commutative monoid; or HR/G for some ring R with $G < GL_1(R)$. Then $\Omega_L BX \cong X$.*

Remark 4.15. In the case of $X = HM$ this is really already a result of [Seg74]. The point there is just that BHM is the Γ-category $\mathcal{F}in_* \to \mathrm{Cat}$ describing the symmetric monoidal category BM (the symmetric monoidal-ness comes from the fact that M is commutative). It's a classical result that endomorphisms of the unique object of this category recover M.

Question 4.16. There are a lot of natural questions to be asked about this going forward:

1. The above theorem is proven by tediously computing with simplicial sets and Γ-sets. Is there a clear categorical or universal reason that it should hold?

2. Are there other interesting mathematical structures that can be encoded as $\mathbb{F}_1$-modules? The first that occur to me as possibilities are matroids and combinatorial species. Do they admit deloopings?

3. Is there a larger class of $\mathbb{F}_1$-modules for which this statement holds? Possibly all of them? It would be particularly interesting to understand deloopings of Durov's generalized rings, also known as Lawvere theories. For instance, algebraic theories have classifying topoi. Do these have any relation the delooping of the associated theory in the above structure?

4. If we want to further deloop $\mathbb{F}_1$-modules, it seems clear that we'd need to involve higher categorical analogues of simplicial sets. In other words, simplicial sets are a great model for $(\infty, 1)$-categories, which makes them a suitable place to build one-fold deloopings of monoids, but higher deloopings of commutative monoids will need to land in some model of (∞, n)-categories (e.g. complicial sets). The idea here is that n-simplices in $sSet_*$ don't have an inherent "direction." As a result, BBM would lose the directional information of M and when we looped back down, we'd end up with the group completion of M. This of course raises the question of what precisely "looping" should be. It seems likely that the correct notion is to take the $(\infty, n-1)$-category of endomorphisms of the base point (deloopings will always be pointed).

5. In [CC20] Connes and Consani give a formulation for taking the homology of an arbitrary simplicial set (i.e. not necessarily a Kan complex) with coefficients in an $\mathbb{F}_1$-module. It is then natural to ask how, if at all, some notion of "homotopy classes of maps" $K \to BA$, for K a simplicial set and A an $\mathbb{F}_1$-module, is related to the homology of K with coefficients in A. At a more basic level, it would be interesting to do some computations in this setting, or either homology or cohomology. It is maybe relevant to point out that they also provide a notion of "homotopy sets" and so forth, so that (to some degree) it makes sense to talk about "homotopy classes of maps." This seems like it might have applications to directed homotopy theory (e.g. taking homology of directed 1-types with coefficients in monoids).

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