Descent Cohomology and Twisted Forms in Homotopy Theory

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- Descent theory answers the question: What information do we need to study R-modules using S-modules?
- The answer is: descent data. Descent data lets us "descend" information about S-modules to information about R-modules.
- If we apply $Spec(-)$ to our rings and think of our modules as sheaves, then descent data manifests as "gluing data."

Definition

A descent datum for a morphism of commutative rings $\phi : R \to S$ consists of:

- *an S-module M*
- *an isomorphism (of $S \otimes_R S$-modules), $\theta : p_0^*(M) \xrightarrow{\cong} p_1^*(M)$*
- *a commutative diagram which constitutes the cocycle condition:*

$$\begin{tikzcd}
& p_{10}^*(M) \cong p_{02}^*(M) \arrow[dr, "p_2^*(\theta)"] & \\
p_{00}^*(M) \cong p_{01}^*(M) \arrow[ur, "p_0^*(\theta)"] \arrow[rr, "p_1^*(\theta)"] & & p_{11}^*(M) \cong p_{12}^*(M)
\end{tikzcd}$$

There is a *category* of descent data.
It is the limit of the diagram:

$$\begin{tikzcd} SMod \arrow[r, shift left] \arrow[r, shift right] & (S \otimes_R S)Mod \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & (S \otimes_R S \otimes_R S)Mod \end{tikzcd}$$

This diagram is just $(-)Mod$ applied to the first three levels of the *Amitsur complex* $S/R^{\bullet}$.

$$\begin{tikzcd}
\vdots & & \vdots \\
S \otimes_R S \otimes_R S & & (S \otimes_R S \otimes_R S)Mod \\
S \otimes_R S \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & \overset{(-)Mod}{\rightsquigarrow} & (S \otimes_R S)Mod \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \\
S \arrow[u, shift left] \arrow[u, shift right] & & SMod \arrow[u, shift left] \arrow[u, shift right] \\
R \arrow[u] & & RMod \arrow[u]
\end{tikzcd}$$

When does the category of descent data tell us about the category of R-modules?

Definition

A morphism of rings $R \to S$ is said to be a *descent* morphism if the functor $RMod \to \lim Mod(S/R^{\bullet})$ is fully faithful.

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A morphism of rings $R \to S$ is said to be an *effective descent* morphism if the functor $RMod \to \lim Mod(S/R^\bullet)$ is an equivalence.

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Example

Grothendieck showed that faithfully flat morphisms of commutative rings are of effective descent.

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Definition

Let $\phi : R \to S$ be an effective descent morphism , and N an R-module. Then a twisted form for N along ϕ is an R-module N' such that $N' \otimes_R S \cong N \otimes_R S$.

There is a set of twisted forms, denote it $Tw_\phi(N)$.

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Theorem (See e.g. Waterhouse)

If ϕ is an effective descent morphism then:

$$Tw_\phi(N) \cong Desc_\phi(N \otimes_R S)$$

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Definition

For an R-module N, define $Aut(N) : CRng^{\backslash R} \to Group$ *by* $Aut(N)(S) = Aut_S(S \otimes_R N)$.

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Theorem (Ibid.)

The set of twisted forms for N along $\phi : R \to S$ is in bijection with the first (non-abelian) cohomology of the cosimplical group $Aut(N)(R/S^\bullet)$:

$$Aut(N)(S) \rightrightarrows Aut(N)(S \otimes_R S) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} Aut(N)(S \otimes_R S \otimes_R S) \cdots$$

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We often call the above cohomology group the *descent cohomology* of N.

Sketch Proof:

- The module $p_0^*(N) = (N \otimes_R S) \otimes_S (S \otimes_R S) \cong N \otimes_R S \otimes_R S$ supports a canonical descent datum given by twisting the S factors:

$$can : p_0^*(N \otimes_R S) \xrightarrow{\cong} p_1^*(N \otimes_R S).$$

- Any *other* descent datum ϕ gives an automorphism after inverting *can*,

$$can^{-1} \circ \phi : p_0^*(N \otimes_R S) \xrightarrow{\cong} p_0^*(N \otimes_R S).$$

- A suitable automorphism can be composed with the canonical descent datum to obtain a new descent datum.

$\square$

To homotopy theory...

Definition (Lurie)

For $\phi : R \to S$, a map of $\mathbb{E}_\infty$-ring spectra, the ∞-category of descent data for ϕ is the totalization of the cosimplicial ∞-category (again based on the Amitsur complex):

$$SMod \rightrightarrows (S \otimes_R S)Mod \to\!\!\!\!\!\!\!\rightrightarrows (S \otimes_R S \otimes_R S)Mod \rightrightarrows\!\!\!\!\!\!\!\rightrightarrows \cdots$$

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Homotopical descent data can be given as an invertible 1-cell and a sequence of higher homotopy cocycle conditions:

Definition

Under the assumptions given above, a descent datum for $\phi : R \to S$ is:

- *an S-module M,*
- *an invertible 1-cell $\theta : p_0^*(M) \to p_1^*(M)$,*
- *a 2-cell*

$$\begin{tikzcd}
& p^*_{10}(M) \simeq p^*_{02}(M) \arrow[dr, "p_2^*(\theta)"] \arrow[d, phantom, "\Updownarrow"] & \\
p^*_{00}(M) \simeq p^*_{01}(M) \arrow[ur, "p_0^*(\theta)"] \arrow[rr, "p_1^*(\theta)"] & {} & p^*_{11}(M) \simeq p^*_{12}(M)
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- *higher n-cells satisfying higher cocycle conditions...*

Definition

For a morphism of $\mathbb{E}_\infty$-ring spectra $\phi : R \to S$, and an R-module N, the space of twisted forms of N is the homotopy limit of the cospan

$$RMod \xrightarrow{-\otimes_R S} Desc_\phi \xleftarrow{N \otimes_R S} *.$$

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Example

Along the morphism $\mathbb{S} \to MU$, $\Sigma^\infty_+ BU$ is a twisted form of MU, as evidenced by the Thom isomorphism

$$MU \wedge \Sigma^\infty_+ BU \simeq MU \wedge MU.$$

Remark

- This generalizes the descent 2-category for 2-categorical descent as studied by Ross Street, Claudio Hermida, Lawrence Breen, and others.

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- Kathryn Hess describes descent data as a category of comodules over a comonad. Her theory, if translated into the language of ∞-categories, is equivalent to this one.

Theorem (Riehl, Verity)

For a homotopy coherent monad of quasicategories $T : \mathcal{C} \to \mathcal{C}$, the quasicategory of descent data is the homotopy limit of the cosimplicial diagram of quasicategories:

$$\begin{tikzcd} T\mathcal{C} \arrow[r, shift left] \arrow[r, shift right] & TT\mathcal{C} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & TTT\mathcal{C} \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \cdots \end{tikzcd}$$

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Remark

For us, the homotopy coherent monad of interest is the extension-of-scalars monad associated to $\phi : R \to S$.

Theorem (B.)

Let $\phi : R \to S$ be a morphism of $\mathbb{E}_\infty$-ring spectra which is of effective descent. Then for an R-module N, there is an equivalence of ∞-categories:

$$Tw_\phi(N) \simeq Desc_\phi(N \otimes_R S).$$

Proof.

Compare these two homotopy pullback diagrams of quasicategories. Since ϕ is of effective descent, $RMod \simeq Desc_\phi$ so the pullbacks are equivalent.

$$\begin{tikzcd} Tw_\phi(N) \arrow[r] \arrow[d] & * \arrow[d, "N \otimes_R S"] \\ RMod \arrow[r, "- \otimes_R S"] & SMod \end{tikzcd}$$

$$\begin{tikzcd} Desc_\phi(N \otimes_R S) \arrow[r] \arrow[d] & * \arrow[d, "N \otimes_R S"] \\ Desc_\phi \arrow[r, "proj"] & SMod \end{tikzcd}$$

$\square$

- Similarly to the discrete case, we now want to determine the space of twisted forms using some kind of "cohomology."
- However, since there is a space of twisted forms, we need a Bousfield-Kan spectral sequence to get at this information!
- For a given R-module N, the cosimplicial space of interest is:

$$Aut(N)(S) \rightrightarrows Aut(N)(S \otimes_R S) \substack{\rightarrow\\\rightarrow\\\rightarrow} Aut(N)(S \otimes_R S \otimes_R S)$$

- Specifically we want to understand the data in cohomological degree one (i.e. the homotopical analogue of H^1).

Theorem (B.)

For a morphism of $\mathbb{E}_\infty$-ring spectra $\phi : R \to S$ and N an R-module, the set of isomorphism classes of descent data on $N \otimes_R S$ is equivalent to $\pi_0 Tot(BAut(R/S^\bullet))$.

Remark

The Bousfield-Kan spectral sequence converges to the homotopy of the above totalization:

$$\pi^s \pi_t \Rightarrow \pi_{t-s} Tot(BAut(R/S^\bullet))$$

If our spaces are sets (e.g. 0-truncated) then π_{-1} of the totalization recovers the first nonabelian cohomology $H^1(R/S^\bullet; Aut(N))$.

Sketch of proof:

- $Aut(-)$ corresponds to taking based loops of $R/S^\bullet$ with base point the canonical descent datum on $N \otimes_R S$ (compare with Dwyer-Kan classification spaces).
- This forgets all other components and produces a cosimplicial loop space, which admits a delooping by a cosimplicial space.
- π_0 of this delooping is precisely the set of descent data on $N \otimes_R S$.

$\square$

Remark

- If $\phi : R \to S$ is a Galois (or Hopf-Galois) extension in the sense of Rognes, the above construction can be reinterpreted as homotopical Galois cohomology.
- In that case, a descent datum corresponds to an (co)action of a (Hopf-)Galois (algebra) group.
- The spectral sequence is a homotopy (co)fixed points spectral sequence.
- Tyler Lawson and David Gepner are doing computations along these lines for the Galois extensions $KO \to KU$.
- Vesna Stojanoska and Akhil Mathew are also doing these kinds of computations for $TMF \to TMF(n)$.

Thom Spectra

- We can get Thom spectra (e.g. MU, $M\Xi$, $X(n)$, MSp etc.) by taking global sections of bundles defined by functions $f : X \to BGL_1(\mathbb{S})$ for X some space.

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Thom Spectra

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- We can then ask about other bundles which are locally equivalent along the "cover" $\mathbb{S} \to Mf$.
- For example: the trivial bundle $\mathbb{S}[X_+]$ is locally equivalent to Mf, and we call this local equivalence the Thom isomorphism

$$Mf \wedge \mathbb{S}[X_+] \simeq Mf \wedge Mf.$$

- Our machinery can compute *other* twists of such Thom spectra (work in progress).

$\mathbb{G}_n$-actions

- For the (profinite-)Galois extension $L_{K(n)}\mathbb{S} \to L_{K(n)}E_n$ a descent datum corresponds to an $L_{K(n)}$-module M and an action of the Morava stabilizer group $\mathbb{G}_n$ on M.

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- Compare with Goerss and Hopkins' identification of the moduli space of spectra X such that $X \wedge E_n \simeq E_n \wedge E_n$ as $BAut(\mathbb{G}_n)$ (note, this means it's connected!).
- Fully developing this case requires dealing with pro-spectra, see work of Daniel Davis, Gereon Quick, Ethan Devinatz and others for some work in this direction.

Thanks to Andrew Salch, Jack Morava, Kathryn Hess and Tyler Lawson for countless helpful discussions regarding this material.

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