Notes from "Interpretations of the Truncated Picard Spectra of KO and KU," given on October 10, 2022 at the Casa Matematica Oaxaca

Jonathan Beardsley

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1 Preliminaries

Definition 1.1. Let R be a commutative ring spectrum with symmetric monoidal ∞-category of modules $(Mod_R, \otimes_R, R)$. Then we define $Pic(R)$ to be the subcategory of Mod_R spanned by invertible R-modules (with respect to $\otimes_R$) and *equivalences* between them. See [ABG18].

Remark 1.2. Note that $Pic(R)$ is an ∞-groupoid (i.e. a "space") because all of its morphisms are invertible. It has a base point, the point corresponding to R itself. It is also an infinite loop space: it inherits the symmetric monoidal structure of Mod_R (note that the tensor product of invertible R-modules remains invertible) and it is grouplike because every invertible R-module has an inverse.

Definition 1.3. Write $BGL_1(R)$ for the connected component of the base point of $Pic(R)$, i.e. the ∞-groupoid with one object (R itself) and the group of R-module equivalences of R as its morphisms. $BGL_1(R)$ remains an infinite loop space. See [ABG+14].

Notation 1.4. Since $Pic(R)$ is an infinite loop space it has an associated spectrum which we will denote $pic(R)$. Similarly, we will write $bgl_1(R)$ for the spectrum associated to $BGL_1(R)$.

Ansatz 1.5. The infinite loop space $Pic(R)$ represents "twists of R-theory." In other words, given an ∞-groupoid X, homotopy classes of maps $\alpha\colon X \to Pic(R)$ are all possible twists of R (co)homology of X.

Computing α-twisted R-(co)homology of a space X begins by noticing that there is an inclusion (by construction) of symmetric monoidal ∞-categories $Pic(R) \hookrightarrow Mod_R$. If we let $\overline{\alpha}$ denote the composite of α with this inclusion then the α-twisted R-homology of X is theh homotopy groups of the colimit of $\overline{\alpha}$, taken in Mod_R (note that we can't take this colimit in $Pic(R)$ because it's not remotely cocomplete. Indeed, in light of (for instance) [ABG+14], the colimit of $\overline{\alpha}$ very much deserves to be written as $X \otimes_\alpha R$, a sort of twisted tensor product, or twisted R-suspension spectrum. For the twisted cohomology, we have the homotopy groups of the R-module mapping spectrum $Maps_R(X \otimes_\alpha R, R)$. We won't really use these specific constructions, but this is what I mean by "twisted (co)homology."

Example 1.6. In the case that $R = KU$ it is common to talk about twisting K-theory of X by classes in $H^3(X;\mathbb{Z})$. It turns out that there is a map of infinite loop spaces $K(\mathbb{Z}, 3) \to Pic(KU)$ and so one can turn classical twists (e.g. from Dixmier-Douady theory, or associated to bundle gerbes) into the sorts of "homotopical" twists described above. See

As in the above example, classes in degree integral homology are often used to twist K-theory. However in [DK70] it is shown that there is a larger group of possible twists. Specifically, they show that one can twist complex K-theory on X by $H^0(X;\mathbb{Z}/2) \times H^1(X;\mathbb{Z}/2) \times H^3(X;\mathbb{Z})$ and real K-theory by $H^0(X;\mathbb{Z}/8) \times H^1(X;\mathbb{Z}/2) \times H^2(X;\mathbb{Z}/2)$. Importantly however, these two sets are not equipped with the usual product group structures, but a twisted group structure that we will describe later.

We can recover those twists (and slightly more) from a homotopical perspective.

2 Truncated Picard Spectra and K-theory Twists

Definition 2.1. Define two spectra $E_U = pic(KU)[0, 3]$ and $E_O = pic(KO)[0, 2]$. These are ***truncations*** of the Picard spectra $pic(KU)$ and $pic(KO)$ in which the homotopy groups above degree 3 and 2 respectively have been killed. They have underlying spaces $\Omega^\infty E_U \simeq Pic(KU)[0, 3]$ and $\Omega^\infty E_O \simeq Pic(KO)[0, 2]$.

The following is an assemblage of classical results:

Proposition 2.2. *There are equivalences of spaces:*

$$Pic(KU)[0,3] \simeq \mathbb{Z}/2 \times B\mathbb{Z}/2 \times B^3\mathbb{Z}$$
$$Pic(KO)[0,2] \simeq \mathbb{Z}/8 \times B\mathbb{Z}/2 \times B^2\mathbb{Z}/2$$

Corollary 2.3. *There are isomorphisms of sets:*

$$E_U^0(X) \cong H^0(X;\mathbb{Z}/2) \times H^1(X;\mathbb{Z}/2) \times H^3(X;\mathbb{Z})$$
$$E_O^0(X) \cong H^0(X;\mathbb{Z}/8) \times H^1(X;\mathbb{Z}/2) \times H^2(X;\mathbb{Z}/2)$$

It will follow from what we discuss now that the above *cannot* be splittings of infinite loop spaces.

Theorem 2.4. *All possibly non-zero k-invariants of the spectra E_U and E_O are non-trivial. Moreover, these k-invariants induce the following group structures:*

1. *The group structure on $E_U^0(X)$ is given by $(a,b,c)+(a',b',c') = (a+a', b+b', c+c'+\beta(b \cup b'))$, where $\cup$ is the cup product on $H^*(X;\mathbb{Z}/2)$ and β is the Bockstein homomorphism $H^2(X;\mathbb{Z}/2) \to H^3(X;\mathbb{Z})$.*

2. *The group structure on $E_U^0(X)$ is given by $(a,b,c)+(a',b',c') = (a+a', b+b', c+c'+b \cup b')$.*

I won't prove this theorem here. It ends up being quite involved, requiring many computations with the Steenrod algebra as well as constructions and analysis of classical Picard 1-categories.

Corollary 2.5. *The Donovan-Karoubi Brauer real and complex Brauer groups of X are respectively isomorphic to $E_O^0(X)$ and $Tors(E_U^0(X))$ where $Tors$ indicates the torsion subgroup.*

However, the groups $E_O^0(X)$ and $E_U^0(X)$ have several other geometric interpretations. In the following we write $E_{O/U}$ to indicate not a cohomology theory associated to the quotient O/U but rather to discuss both E_O and E_U simultaneously. Throughout we will also write "real/complex." The statements for E_O will correspond to the "real" case and the statements for E_U will correspond to the "complex" case.

Corollary 2.6. *The group $E_{O/U}^0(X)$ is isomorphic to:*

1. *The Brauer group of real/complex $\mathbb{Z}/2$-graded continuous trace C^*-algebras with spectrum X.*

2. *The group of isomorphism classes of $\mathbb{Z}/2$-graded 2-line bundles on X.*

3. *The group of connected components of $\mathfrak{Twist}_{KO/KU}(X)$, the groupoid of K-theory twists described in* [FHT11].

Proof. The first statement follows from [Par88] for the complex case and [Mou14] for the real case. The second follows from [KLW21]. The third from [FHT11]. □

It was already know of course that all of the above structures formed isomorphic groups, but our result shows that these classical sources of K-theory of twists can all be considered homotopy theoretic twists governed by $Pic(KU)$ and $Pic(KO)$.

Lemma 2.7. *Let R be a connective ring spectrum. Then $bgl_1(R[0,n]) \simeq bgl_1(R)[0,n+1]$.*

Proposition 2.8. *When X is connected, any of the equivalent structures described above, i.e. graded C^*-algebras, graded 2-lines and Freed-Hopkins-Teleman twists of K-theory, are equivalent to the group of $ku[0,2]$-line bundles on X in the complex case and $ko[0,1]$-line bundles on X in the real case.*

Proof. This follows from the above lemma, the fact that in general $bgl_1(R)^0(X)$ classifies R-line bundles on X, and the fact that $KU[0,2] \simeq ku[0,2]$ and $KO[0,2] \simeq ko[0,2]$. □

This is moderately mysterious to me, namely that these things from parameterized homotopy theory (i.e. bundles of truncated K-theory spectra) are the same data as very geometric things.

3 Twisted String and Spin Structures

An elementary k-invariant computation proves the following:

Proposition 3.1. *Let F_U be the fiber of the connective cover $BString \to BSO$ and let F_O be the fiber of the connective cover $BSpin \to BO$. Then both F_U and F_O have one possible non-trivial k-invariant, and in both cases it is indeed non-trivial.*

Thus our main theorem leads to the following:

Corollary 3.2. *There are equivalences of infinite loop spaces:*

$$BGL_1(KU)[0,3] \simeq F_U$$
$$BGL_1(KO)[0,2] \simeq F_O$$

From this it follows that lifts of the following kind

$$\begin{array}{ccc} & & BString \\ & \nearrow & \downarrow \\ X & \longrightarrow & BSO \end{array} \qquad\qquad \begin{array}{ccc} & & BSpin \\ & \nearrow & \downarrow \\ X & \longrightarrow & BO \end{array}$$

are torsors for $bgl_1(KU)[0,3]^0(X)$ and $bgl_1(KO)[0,2]^0(X)$ respectively, assuming there is at least one lift. In other words, if X admits such a lift then we can twist that lift (and get every other lift as a twist) by $ku[0,2]$-lines and $ko[0,1]$-lines respectively.

4 The Anderson Dual of $\mathbb{S}$

The final result to state is the relationship between E_U and the Andereson dual of the sphere $I_{\mathbb{Z}}$.

Definition 4.1. Define two functors $Spectra \to Spectra$ by $I^*_{\mathbb{Q}}(X) = Hom(\pi_{-*}X, \mathbb{Q})$ and $I^*_{\mathbb{Q}/\mathbb{Z}}(X) = Hom(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$. As these functors both satisfy Brown representability we write $I_{\mathbb{Q}}$ and $I_{\mathbb{Q}/\mathbb{Z}}$ respectively for their representing spectra. We can then define

$$I_{\mathbb{Z}} = fib(I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}})$$

The Anderson duality functor $Spectra \to Spectra$ is now defined by $X \mapsto Map(X, I_{\mathbb{Z}})$ and so $I_{\mathbb{Z}}$ itself is the ***Anderson dual*** of $\mathbb{S}$.

The spectrum $I_{\mathbb{Z}}$ has the following homotopy groups:

$$\pi_*(I_{\mathbb{Z}}) = \begin{cases} 0 & i > 0 \\ \mathbb{Z} & i = 0 \\ 0 & i = -1 \\ Hom(\pi_{1-i}\mathbb{S}) & i < -1 \end{cases}$$

Progressing downward from the zeroth degree this yields homotopy groups

$$\pi_{-*}I_{\mathbb{Z}} \cong \{\ldots, \mathbb{Z}, 0, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24, \ldots\}$$

The following is essentially a result of the fact that on spectra of finite type the functor $I_{\mathbb{Z}}$ is an auto-equivalence of $Spectra$ and the first three k-invariants of $\mathbb{S}$ are non-trivial:

Proposition 4.2. *The spectrum* $\Sigma^3(I_{\mathbb{Z}}[-3,\infty))$ *is equivalent to* E_U.

In other words, these two spectra have the same homotopy groups and one can compute that their k-invariants are also the same, so they are abstractly equivalent.

One reason this is interesting is that Freed and Hopkins hypothesize the spectrum $I_{\mathbb{Z}}$ to play a role in classifying invertible topological field theories (as well as symmetry protected topological phases, but we won't get into that):

Ansatz 4.3. Invertible extended topological field theories of dimension n with symmetry group H_n are maps of spectra

$$\phi\colon \Sigma^n MTH_n \to \Sigma^n(I_{\mathbb{Z}}[-n,\infty))$$

where MTH_n is the Madsen-Tillman spectrum.

So maps from suitable cobordism spectra into E_U are the same data as invertible extended 2-dimensional field theories (hypothetically). This is somewhat interesting because on one hand E_U seems to "know about" 2-dimensional invertible extended topological field theories, from which one can extract 2-line bundles, and on the other hand, from the preceding sections, E_U is the classifying theory for graded complex 2-line bundles. The "meaning" of all these things is still a mystery to me however.

There is a natural way in which we might "shift up" both sides of the equivalence $E_U \simeq \Sigma^3(I_{\mathbb{Z}}[-3,\infty))$. On one hand we could replace the right hand side with $\Sigma^3(I_{\mathbb{Z}}[-4,\infty))$ which would presumably play a role in 3-dimensional TFTs. On the other side, we would replace the Picard spectrum with the Brauer spectrum $br(KU)$. One can define this imprecisely as the space of Azumaya algebras over KU for some notion of Azumaya algebra spectrum, or one can mimic the Picard spectrum definition:

Definition 4.4. Let Mod_R be the symmetric monoidal ∞-category of R-modules over a commutative ring spectrum R. Write $2Mod_R$ for the ∞-category of modules over Mod_R in the $(\infty,2)$-category of ∞-categories. Inductively define $nMod_R$ to be the ∞-category of modules over $(n-1)Mod_R$ in the (∞,n)-category of $(\infty,n-1)$-categories.

Remark 4.5. There are obviously several refinements that could be made to the above definition, but as a rough notion this will suffice.

Definition 4.6. Define $\mathbb{G}_m^n(R)$, for $n>0$ to be the ∞-group of invertible objects in $nMod_R$. Let $\mathbb{G}_m^0(R)$ denote $\pi_0(\mathbb{G}_m^1(R))$.

Remark 4.7. Note that $\mathbb{G}_m^1(R) = Pic(R)$ and $\mathbb{G}_m^2(R) = Br(R)$. It might be natural to define $\mathbb{G}_m^0(R)$ to be something like the classical Brauer group of $\pi_0(R)$, but this gets the "wrong thing" in the case of KU and KO. For K-theory spectra we would like $\mathbb{G}_m^0(R)$ to be something like "the Brauer group of the thing that we've taken K-theory of." Again, this isn't *quite* right because KU and KO are not the K-theory of $\mathbb{C}$ and $\mathbb{R}$, respectively. It might be that working with condensed (a la Scholze and Clausen) or pyknotic (a la Barwick and Haine) spectra could resolve this issue.

Now we can state the following speculative conjecture:

Conjecture 4.8. *For all $n \geq 0$ there are equivalences $\mathbb{G}_m^n(KU)[0, n+2] \simeq {}_{n+2}I_{\mathbb{Z}}$. Equivalently, there are equivalences $I_{\mathbb{Z}}(\mathbb{G}_m^n(KU)[0, n+2]) \simeq \mathbb{S}[0, n+1]$.*

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