On the PROB of Singular Braids

Jonathan Beardsley (UNR), Suhyeon Lee (Berkeley), Brendan Murphy (UW), Luke Trujillo (Harvey Mudd)

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Let $(C, \otimes, 1_C, \beta)$ denote a braided monoidal category, where β is the symmetry transformation inducing the structure isomorphisms $\beta_{x,y} \colon x \otimes y \xrightarrow{\sim} y \otimes x$ for all $x, y \in C$.
Given any object $x \in C$, we get a system of braid group representations:

$$\rho_1 \colon B_1 \to Aut_C(x) \qquad e \mapsto (id_x \colon x \to x)$$

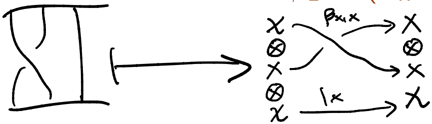
$$\rho_2 \colon B_2 \to Aut_C(x^{\otimes 2}) \qquad 1 \mapsto (\beta_{x,x} \colon x \otimes x \to x \otimes x)$$

$$\rho_3 \colon B_3 \to Aut_C(x^{\otimes 3}) \qquad \beta_1 \mapsto (\beta_{x,x} \otimes id_x \colon x^{\otimes 3} \to x^{\otimes 3})$$

$$\beta_2 \mapsto (id_x \otimes \beta_{x,x} \colon x^{\otimes 3} \to x^{\otimes 3})$$

$$\rho_4 \colon B_4 \to Aut_C(x^{\otimes 4}) \qquad \beta_1 \mapsto (\beta_{x,x} \otimes id_{x^{\otimes 2}} \colon x^{\otimes 4} \to x^{\otimes 4})$$

$$\beta_2 \mapsto (id_x \otimes \beta_{x,x} \otimes id_x \colon x^{\otimes 4} \to x^{\otimes 4})$$



$\vdots$

This manifests in the following equivalence of categories due to Joyal and Street:

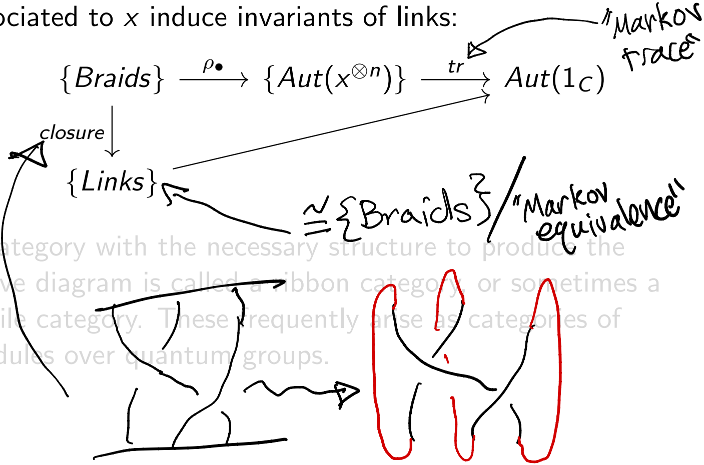
$$BMC(\mathbb{B}, C) \simeq Cat(1, C) \simeq C$$

where BMC is the 2-category of braided monoidal categories and braided monoidal functors, and $\mathbb{B}$ is the category defined in the following way:

$$obj(\mathbb{B}) = \mathbb{N}$$
$$\mathbb{B}(n, m) = \begin{cases} B_n & n = m \\ \varnothing & n \neq m \end{cases}$$

In other words, $\mathbb{B}$ is the free braided monoidal category on 1.

If our category C satisfies some additional conditions (e.g. having a suitable trace operator $tr\colon Aut(x^{\otimes n}) \to Aut(1_c)$) then the braid representations associated to x induce invariants of links:



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$$\begin{array}{ccccc} \{Braids\} & \xrightarrow{\rho_\bullet} & \{Aut(x^{\otimes n})\} & \xrightarrow{tr} & Aut(1_C) \\ \downarrow{\scriptstyle closure} & & & \nearrow & \\ \{Links\} & & & & \end{array}$$

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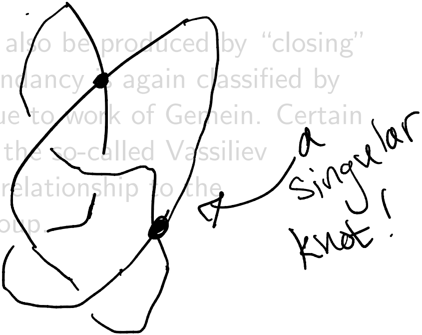
Goal

We would like to mimic this procedure to produce invariants of *singular* knots and links, i.e. knots and links in which we allow a finite number of transverse self-intersections.

Remark

Singular knots and links can also be produced by "closing" singular braids, and the redundancy is again classified by "singular Markov moves," due to work of Gemein. Certain invariants of singular braids, the so-called Vassiliev invariants, have a surprising relationship to the Grothendieck-Teichmüller group.



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Definition

Define the singular braid monoid on n-strands, denoted SB_n, to be the monoid generated by the symbols b_i, b_i^{-1}, s_i for $i = 1, 2, \ldots, n-1$ subject to the following relations:

$$b_i b_j = b_j b_i, \text{ if } |i-j| > 1, \tag{1}$$

$$s_i s_j = s_j s_i, \text{ if } |i-j| > 1, \tag{2}$$

$$s_i b_j = b_j s_i, \text{ for all } 0 < i, j < n, \tag{3}$$

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \tag{4}$$

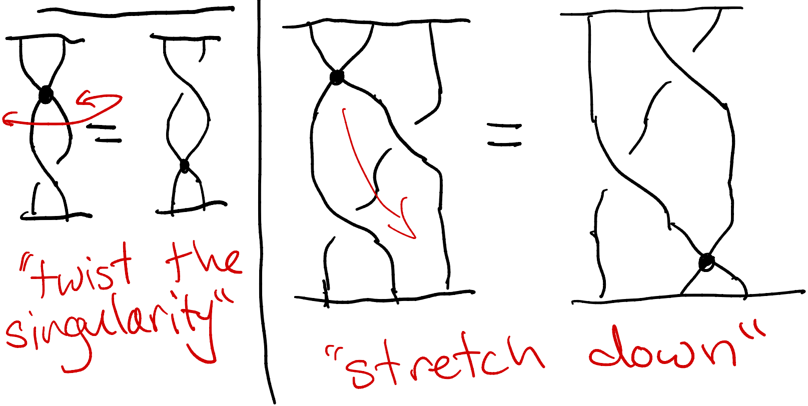
$$b_i b_{i+1} s_i = s_{i+1} b_i b_{i+1}, \tag{5}$$

$$s_i b_{i+1} b_i = b_{i+1} b_i s_{i+1}, \tag{6}$$

$$b_i b_i^{-1} = b_i^{-1} b_i = 1. \tag{7}$$

Some pictures of the singular braid relations:

$s_1 b_1 = b_1 s_1$ | $s_1 b_2 b_1 = b_2 b_1 s_2$



Goal

Just as a choice of object in a braided monoidal category induces a system of representations of the braid groups, determine a structure on categories such that a choice of object induces a system of representations of the singular braid monoids.

$$SB_n \xrightarrow{\rho_n} \mathrm{End}_C(X^{\otimes n})$$

Problem

Unfortunately, for several reasons, this structure cannot be operadic in nature. In particular, it does not make sense to define a "singularly braided monoidal category."

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$s_i s_{i+1} s_i \neq s_{i+1} s_i s_{i+1} \Rightarrow S$ "symmetry" cannot be a natural transform!

One solution to this problem is generalize from operads to PROPs, which can be thought of as operads with operations having multiple outputs and inputs (and are even more general than properads). In fact it will be useful to consider a more general (but less frequently studied) class of objects, PROBs.

Definition

A PROB (resp. PROP) is a braided (resp. symmetric) monoidal category whose objects are in bijection with the set $\mathbb{N}$ *and whose monoidal structure corresponds to addition of natural numbers.*

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Joyal and Street's category $\mathbb{B}$ is a PROB (in fact it is the initial PROB).

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Remark

- In a PROP or PROB P, the set $P(n, m)$ is the set of operations with n inputs and m ouputs.
- The category of braided (resp. symmetric) operads embeds fully faithfully into the category of PROBs (resp. PROPs).
- An *algebra* over a PROB P is exactly a braided monoidal functor $P \to C$ for some braided monoidal category C.
- The term PROP is originally due to Mac Lane and stands for "PROduct and Permutation." In PROB, the permutations are replaced by braid group actions.

Our primary example is the following PROB of singular braids:

Definition

Let $\mathbb{SB}$ be the category defined in the following way:

$$obj(\mathbb{SB}) = \mathbb{N}$$

$$\mathbb{SB}(n, m) = \begin{cases} SB_n & n = m \\ \varnothing & n \neq m \end{cases}$$

Theorem (BLMT)

Let C be a braided monoidal category. Then the set of $\mathbb{SB}$-algebras in C is in bijection with the set of pairs $(x \in C, \rho_\bullet)$ where $\rho_\bullet \colon SB_\bullet \to End_C(x^{\otimes \bullet})$ is a system of monoid morphisms that respects juxtaposition of singular braids.

"generated" under juxtaposition and composition by $s_1, b_1, b_1^{-1} \in SB_2$

Conjecture

If C is additionally a ribbon category, then every $\mathbb{SB}$*-algebra A in C induces an isotopy invariant*

$$S_A \colon \{Singular\ Links\} \to End_C(1_C).$$

Question

Is there a braided operad $\mathcal{SB}$ *such that* $\mathbb{SB}$ *is the free PROB generated by* $\mathcal{SB}$*?*

one 0-ary operation x

and one unary operation

$x \otimes x \longrightarrow x \otimes x$?

Goal

Given a quantum group (or quasi-triangular Hopf-algebra) H and an H-module V, there is an R-matrix $R_V \in Aut_{Mod_H}(V^{\otimes 2})$ determining a braided monoidal functor $F_V \colon \mathbb{B} \to Mod_H$ taking n to $V^{\otimes n}$. In light of the preceding conjecture,determining an additional matrix $S_V \in End_{Mod_H}(V^{\otimes 2})$ which lifts F_V to $\mathbb{SB}$, e.g. satisfying $S_V R_V = R_V S_V$, is effectively a linear algebra problem over H. Finding solutions to the relevant systems of equations appears to be tractable by using programs like CoCalc.

Example

The Jones polynomial, a well-known invariant of knots and links, can be obtained from the following R-matrix acting on the free rank 2 module over the ring $\mathbb{C}[t^{1/2}, t^{-1/2}]$:

$$\begin{bmatrix} -t^{1/2} & 0 & 0 & 0 \\ 0 & -t^{1/2}+t^{3/2} & -t & 0 \\ 0 & -t & 0 & 0 \\ 0 & 0 & 0 & -t^{1/2} \end{bmatrix}$$

If q and p are any Laurent polynomials in the variable $t^{-1/2}$, then the following matrix extends the braid group representation to a singular braid monoid representation:

$$\begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p+tq & -t^{1/2}q & 0 \\ 0 & -t^{1/2}q & p+q & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$

not nec. invertible

Thank you!