A Very Brief Introduction to Stable Homotopy Theory

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Invariants and Classification

Many problems in mathematics are, at their core, problems of classification. Roughly, given some class of mathematical objects $\mathfrak{C}$ equipped with a notion of "isomorphism," we'd like to answer the following question:

1. Can we produce a complete set of invariants for $\mathfrak{C}$, i.e. given two objects $X, Y \in \mathfrak{C}$, two collections of data $\{X_i\}$ and $\{Y_i\}$ such that $X \cong Y$ if and only if $X_i \cong Y_i$ for all i?

Of course one can always trivially take X and Y themselves to be the "invariants," but nothing is gained by doing this! In general we hope that the sets of invariants are in some way *simpler* than the objects of $\mathfrak{C}$. Some classes of objects that have been fully classified in this way are, for instance: 2-dimensional closed manifolds; algebraic surfaces of 2 (complex) dimensions; finite simple groups; simple Lie algebras.

At its heart, algebraic topology has the same goal, i.e. to use invariants to classify topological spaces with respect to some notion of isomorphism between them. Initially one might attempt to classify topological spaces "up to homeomorphism," but considering how complicated topological spaces are, this is most likely forever beyond the reach of human beings. One might then try to weaken the notion of isomorphism being used and ask to classify topological spaces "up to homotopy." As far as we can tell, this is still effectively impossible, but we can at least make a start!

Roughly, the idea of algebraic topology is to produce *algebraic homotopy invariants* of topological spaces. This is readily encoded using the language of category theory. That is to say, an algebraic topologist's goal is to produce, study, and compute the outputs of *functors* $F\colon Top \to \mathbb{A}$ which give isomorphic output for homotopy equivalent topological spaces, where $\mathbb{A}$ is *some* category of "algebraic objects," e.g. groups, rings, sets, or graded versions of these things. For technical reasons, and to avoid telling more lies than necessary, we will consider invariants of *pointed spaces* in this talk, i.e. functors $F\colon Top_* \to \mathbb{A}$ which are invariant up to *pointed homotopy*, but the distinction is not critical to understanding what follows.

Remark. An important thing to remember about invariants of any kind is that *complete* invariants are quite rare. In other words, *usually* we can use invariants to tell objects apart, $F(X) \ncong F(Y) \Rightarrow X \ncong Y$, but it's not necessarily true that $F(X) \cong F(Y) \Rightarrow X \cong Y$.

You already know some examples of algebraic homotopy invariants of topological spaces:

1. For each n, and any Abelian group A, we have an invariant $H_n(-;A)\colon Top_* \to Ab$, which we can take to be either (reduced) singular or cellular homology with coefficients in A. For fixed A, we typically package these invariants into a single functor $H_*(-,A)\colon Top_* \to GrAb$ with codomain the category of *graded* abelian groups.

2. Similarly, for any abelian group A, we can consider the graded abelian *co*homology groups $H^*(-,A)\colon Top_*^{op} \to GrAb$. If A is in fact a ring, we get lucky with cohomology because it gives us more than just graded abelian groups, it gives a functor to $GrRng$, the category of graded rings (using the cup product).

3. Given a pointed space X, we can produce a ***pointed set*** $\pi_0(X)$ which is the set of connected components of X (where the base point of $\pi_0(X)$ is the connected component of the base point in X).

4. More generally, we can define an invariant $\pi_*\colon Top_* \to GrSet$ by the formula $\pi_n(X) := Map_*(S^n, X)/\{\text{homotopy}\}$. Again we get lucky because when $i > 0$, the set $\pi_i(X)$ is a group, and even better, when $i > 1$, $\pi_i(X)$ is an abelian group! As a result, we often talk about the ***homotopy groups of the space*** X (where the base point of X is typically left implicit). You may be familiar with the case that $i = 1$, which is the ***fundamental group*** of X.

Remark. In general, we like to have invariants that have a lot of structure, because we can use that structure to tell them apart. For instance, using cohomology with coefficients in a ring, we can often produce isomorphic Abelian groups which are not isomorphic *as rings.* Producing invariants with more and more structure (which is still homotopy invariant) is a recurring theme in algebraic topology (cf. cohomology operations, homology cöoperations

All of the above invariants respect homotopy equivalence, but *none* of them are complete invariants (in fact, we don't have any complete invariants of spaces-up-to-homotopy besides the trivial ones). However, they all have nice properties that seem like reasonable things to ask of homotopy invariants, e.g. they take direct sums of spaces to direct sums of algebraic objects; when we have a fiber sequence of spaces we get a long exact sequence of algebraic objects, and this makes them manageable, and importantly, sometimes even computable! In a certain sense, homotopy groups are the *finest* of all such invariants, in that if two spaces look the same to homotopy groups then they will always look the same to every other invariant (I'm fudging the details quite a bit here, but it's morally true). Unfortunately, homotopy groups are *incredibly difficult* to compute. We don't even know what $\pi_*(S^n)$ is for all spheres S^n.

As a result of homotopy groups being both powerful and very difficult to compute, much of algebraic topology has been directed toward the following goals:

- Compute the homotopy groups of spaces.
- If we're unable to do that, find a simpler invariant that we *can* compute, and see if we can leverage it to give us information about homotopy groups.

Remark. There are MANY more invariants than the examples above (in fact, probably about $2^{2^{\aleph_0}}$). Some things to look up if you're interested are:

- topological K-theory,
- cobordism theories,
- stable homotopy groups,
- elliptic cohomology (connections to number theory!),
- Brown-Peterson cohomology.

Representability and Eilenberg-MacLane Spaces

One direction of research in modern algebraic topology is to study the invariants themselves rather than explicitly studying topological spaces (although all of this work is still ultimately grounded in getting a better understanding of topological spaces). One can think of this as doing something like "algebra parameterized by the category of pointed spaces." In other words, instead of studying, say, graded Abelian groups, we study functors $H\colon Top_* \to GrAb$, so we are studying "families" of graded abelian groups. Indeed, many of the standard constructions in abstract algebra, e.g. rings, modules, tensor products, exact sequences, can be extended to this "parameterized" world.

We'll come back to "parameterized algebra" later, but first lets examine a property that a functor may possess that makes it easier to understand: representability. Given a functor between two categories $F\colon C \to D$, we can ask that for any $c \in C$, we have a (functorial, natural) isomorphism $F(c) \cong Hom_C(c,d)$ for some fixed $d \in C$. In other words, F is *represented* by the object d. This is a particularly useful property because we can sometimes replace studying the functor F itself with studying the object d. And this happens in a crucial example in algebraic topology.

Theorem. *Let X be a pointed topological space and A an Abelian group. Then there is a sequence of pointed spaces, denoted $\{B^n A\}$, such that*

$$H^n(X,A) \cong Map_*(X, B^n A)/\{\text{homotopy}\}$$

These spaces are called Eilenberg-MacLane spaces, and they have a number of special properties:

- $B^0 A \simeq A$, and $B^1 A$ is the so-called "classifying space" for A, i.e. the space with the property that for any other space X, homotopy classes of maps $X \to BA$ are in bijection with principal A-bundles on X.

- The homotopy groups of B^nA are entirely concentrated in degree n, and in that degree they are isomorphic to A itself, i.e. $\pi_n(B^nA) \cong A$ and $\pi_i(B^nA) \cong 0$ for any $i \neq n$.
- The set $Map_*(S^1, B^nA)$ has a natural topology on it and with this topology $Map_*(S^1, B^nA) \simeq B^{n-1}A$.

Example. If we take $A = \mathbb{Z}$ then we have $A \cong B^0A \cong \mathbb{Z}$, $B^1A \cong S^1$, $B^2A \cong \mathbb{C}P^\infty$, and then other spaces which don't have special names. It may not be surprising that $B^1\mathbb{Z} \cong S^1$ if you consider that you want a space with the property that "loops" in it recover the integers.

Spectra

It would require too much technical background to state here, but there is a famous theorem called the ***Brown Representability Theorem*** which implies, more or less, that ***every*** homotopy invariant of spaces which behaves, more or less, like singular cohomology (this is given by an explicit list of conditions) is representable in the above way, i.e. as a sequence of spaces. This leads us to studying the sequences of spaces that arise in this way, and we call such sequences ***spectra***. A related concept that you might want to Google is that of an "infinite loop space," which is roughly the same data (Question: can you figure out from the previous section why such things might be called infinite loop spaces?).

In particular, there is a category Sp of spectra which has several nice properties:

- There is a well behaved functor from pointed topological spaces into spectra: $\Sigma^\infty\colon\ Top_* \to \mathrm{Sp}$.
- There is a well behaved functor from abelian groups into spectra: $H\colon\ Ab \to \mathrm{Sp}$.
- Given two spectra E and E', there is a notion of "tensor product" so that $E \otimes E'$ is another spectrum, and a notion of "internal hom" so that $Hom_{\mathrm{Sp}}(E, E')$ is another spectrum.
- There is a "homotopy groups" functor $\pi_*^S\colon\ \mathrm{Sp} \to GrAb$ which behaves much like the homotopy groups functor described above.

Crucially, every spectrum E gives a covariant "homology" functor $E_*\colon\ Top_* \to GrAb$ and a contravariant "cohomology" functor $E^*\colon\ Top_*^{op} \to GrAb$ (and by Brown Representability, every suitable homology and cohomology functor gives a spectrum) by the following formulae:

$$E^*(X) := \pi_*^S(Hom_{\mathrm{Sp}}(\Sigma^\infty X, E))$$

$$E_*(X) := \pi_*^S(\Sigma^\infty X \otimes E)$$

In particular, by taking $E = HA$ for some Abelian group A, we have $HA^*(X) \cong H^*(X, A)$ and $HA_*(X) \cong H_*(X, A)$. In other words, in a certain sense, homology and cohomology are both "representable," and so we can study the spectrum HA as an object in its own right to better understand the invariants we obtain by taking $HA^*(-)$ and $HA_*(-)$ of various spaces.

My Work

In my research, for the most part, I study spectra. This often means doing category theory! One of the nice facts about spectra is that they behave *a lot* like Abelian groups (indeed, spectra can be thought of as a homotopy theoretic "enhancement" of the category of Abelian groups in a very real way). As a result we can talk about *ring* spectra, and *module* spectra which generalize the classical notions of modules and rings. Doing this kind of algebra with spectra has come to be called derived algebra, higher algebra, or "brave new algebra." Particular interests of mine in this setting are: Galois theory, descent theory, modules and comodules over Hopf algebras (this allows me to say especially scary things like "spectral quantum group"), and deformation theory.

Some References

There are a lot of books out there on algebraic topology and homotopy theory. The canonical starting point is Hatcher's book *Algebraic Topology*, with which I expect most of you are familiar. There are lots of other references that can be used instead of Hatcher's book. What's more important for our purposes is finding references for *after* Hatcher. Luckily there are a few good ones that are freely available on the internet:

- Davis and Kirk's book *Lecture Notes and Algebraic Topology*, available here: www.maths.ed.ac.uk/~v1ranick/papers/davkir.pdf,
- May's *A Concise Course in Algebraic Topology*, available here: www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf and May and Ponto's *More Concise Algebraic Topology*, available here: www.math.uchicago.edu/~may/TEAK/KateBookFinal.pdf,
- Adams' *Stable Homotopy and Generalised Homology*, available here: people.math.rochester.edu/faculty/doug/otherpapers/Adams-SHGH.pdf.