Hopf-Galois Extensions and $\mathbb{E}_k$-bialgebras in Spectra

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March 28, 2015

Galois Extensions

- A Galois extension is a map of rings $f : A \to B$, such that a group G acts on B and B “locally” looks like G:

$$B \otimes_A B \cong B \otimes G \cong \prod_{g \in G} B$$

- We also have that A can be recovered from B by doing Galois descent (taking fixed points!):

$$A = \{b \in B : gb = b\}$$

Hopf-Galois Extensions

- A *Hopf-Galois* extension is a map of rings $f : A \to B$, such that a bialgebra H coacts on B and B "locally" looks like H:

$$B \otimes_A B \cong B \otimes H$$

- We also have that A can be recovered from B by taking *co*fixed points:

$$A = \{b \in B : c(b) = b \otimes 1\} \cong \lim(B \overset{c}{\rightrightarrows} B \otimes H)$$

- If $A \to B$ is a G-Galois extension for finite G, $A \to B$ is a Hopf-Galois extension for the Hopf-algebra $Hom_A(A[G], B)$.

Rognes has defined both of the above for $\mathbb{E}_\infty$-ring spectra and commutative/cocommutative Hopf-algebras.

Example

- $KO \to KU$ is a Galois extension with Galois group C_2.
- For finite subgroups $K < \mathbb{G}_n$ of the Morava stabilizer group, $L_{K(n)}\mathbb{S} \to E_n^{hK}$ is a K-Galois extension.
- $\mathbb{S} \to MU$ is a Hopf-Galois extension with bialgebra $\Sigma^\infty_+ BU$.
- One can show that $MSU \to MU$ is a Hopf-Galois extension with associated bialgebra $\Sigma^\infty_+ \mathbb{C}P^\infty$.

Computations indicate that other naturally occurring Thom spectra should be Hopf-Galois extensions of $\mathbb{S}$, but their associated bialgebras aren't $\mathbb{E}_\infty$.

Example

- $\mathbb{S} \to H\mathbb{Z}$ with bialgebra $\Sigma^\infty_+ \Omega^2 S^3\langle 3\rangle$.
- $\mathbb{S}^\wedge_2 \to H\mathbb{Z}/2$ with bialgebra $\Sigma^\infty_+ \Omega^2 S^3$.
- $\mathbb{S} \to X(n) = Th(\Omega SU(n))$ with bialgebra $\Sigma^\infty_+ \Omega SU(n)$.
- $X(n) \to X(n+k)$, with bialgebra $\Sigma^\infty_+ \Omega V_k(\mathbb{C}^{n+k})$ (loops on the Stiefel manifold of orthonormal k-frames in $\mathbb{C}^{n+k}$.

How can we talk about $\mathbb{E}_n$-coalgebras in the category of spectra?

Idea:

- $\mathbb{E}_n$-coalgebras of $\mathcal{S}$ should be $\mathbb{E}_n$-algebras of $\mathcal{S}^{op}$.
- There exists a symmetric monoidal structure on $\mathcal{S}^{op}$ (given explicitly by Barwick, Glasman and Nardin) which we can restrict to an $\mathbb{E}_n$-monoidal structure to get $\mathbb{E}_n$-algebras.

Definition

$\mathbb{E}_n$-coalgebras in $\mathcal{S}$ are $\mathbb{E}_n$-algebras in $\mathcal{S}^{op}$ using the Barwick-Glasman-Nardin construction.

Remark (Advantages of BGN Construction)

- We can generalize immediately to $\mathbb{E}_n$-coalgebras over an $\mathbb{E}_k$-ring spectrum R as long as $n \leq k$.
- Strict $\mathbb{E}_n$-monoidal functors preserve coalgebras.
- There is a tensor product on coalgebras which is computed in the underlying category.
- We can easily produce categories of comodules over a given coalgebra.

Example

- Every space in the ∞-category of topological spaces (Kan complexes) is an $\mathbb{E}_\infty$-coalgebra.
- From above, it follows that the suspension spectrum, $\Sigma_+^\infty X$ of any space X is an $\mathbb{E}_\infty$-coalgebra in the ∞-category of spectra.
- Suppose R is an $\mathbb{E}_n$-ring spectrum. Then B^nR, the n-fold delooping of R, is an $\mathbb{E}_n$-coalgebra in spectra.

How then can we talk about *bialgebras* in ∞-categories?

Idea:

- The category of spectra, $\mathcal{S}$, is symmetric (i.e. $\mathbb{E}_\infty$) monoidal, so the category of $\mathbb{E}_n$-coalgebras is symmetric monoidal.
- We can discuss $\mathbb{E}_m$-algebras in $\mathbb{E}_n$-coalgebras, thence: $\mathbb{E}_m$-*co*$\mathbb{E}_n$-bialgebras in $\mathcal{S}$.

Remark

By defining bialgebras as algebras *inside* a category of coalgebras we get compatibility of the multiplication and comultiplication for free (cf. group schemes).

Example

- If $X \simeq \Omega^n Y$ is an n-fold loop space then $\Sigma^\infty_+ X$ is a cocommutative $\mathbb{E}_n$-bialgebra in spectra.
- If R is an $\mathbb{E}_n$-ring spectrum then $B^m R$ is an $\mathbb{E}_{n-m}$-$co\mathbb{E}_m$-bialgebra in spectra.

Theorem (B.)

Let $f : X \to BGL_1(\mathbb{S})$ be an n-fold loop map. Then the Thom spectrum Mf is a comodule over the cocommutative $\mathbb{E}_n$-bialgebra $\Sigma^\infty_+ X$.

Corollary

If the composition $X \xrightarrow{f} BGL_1(\mathbb{S}) \to BGL_1(H\mathbb{Z})$ is null (i.e. Mf is $H\mathbb{Z}$-oriented), then the cotensor product $Mf \square_{\Sigma^\infty_+ X} \mathbb{S}$ is homotopy equivalent to $\mathbb{S}$ (i.e. convergence of the Mf-based Adams-Novikov spectral sequence).

Remark

In other words, $\mathbb{S} \to Mf$ is a Hopf-Galois extension with associated bialgebra $\Sigma^\infty_+ X$. This corollary essentially follows from work of Bousfield.

Further Work:

- Cases like the iterated Hopf-Galois extensions $X(n) \to X(n+1) \to X(n+2)$ suggest a Galois correspondence in certain cases. Can we produce at least one direction of such a correspondence more generally by filtering bialgebras (in this case we filtered $\Sigma^{\infty}_{+} BU$)?
- The Brown-Gitler spectra interpolate between $\mathbb{S}_{(2)}$ and $H\mathbb{Z}/2$, but are not ring spectra. Similarly, there are interpolative (non-ring) spectra between the $X(n)$-spectra. One can "descend" these sequences of comodules by taking cofixed points. Relevant definitions do not yet exist to talk about such "extensions of comodules".
- The $X(n)$ spectra are central to the proof of Nilpotence Theorem of Devinatz, Hopkins and Smith. Can the above algebro-geometric considerations shed light on this proof?

- Given that we can produce Thom spectra from parameterized spectra, how much of this structure exists on that level? What about on the level of locally constant sheaves of spectra? Are Hopf-Galois extensions actually manifestations of more general structure inside of a Wirthmüller context?
- What sort of properties do categories of bialgebras have?
- If we allow ourselves to think of $\mathbb{E}_n$-*co*$\mathbb{E}_m$-bialgebras as $\mathbb{E}_m$-monoidal group schemes in $\mathbb{E}_k$-ring spectra, how much Derived Algebraic Geometry can we get away with doing?

Thanks to Andrew Salch, Jack Morava, Kathryn Hess, Clark Barwick and Tyler Lawson for countless helpful discussions regarding this material.