Some Galois Theory For Bordism Homology

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Outline

- describe G-bordism homology of spaces for varying group G
- redefine Galois extensions to allow for a more general class of examples
- G-bordism homology is Galois over stable homotopy

Bordism

compact diff'ble

<u>Def'n:</u> A bordism between n-manifolds M and N is an $n+1$-manifold P s.t. $\partial P = M \sqcup N$.

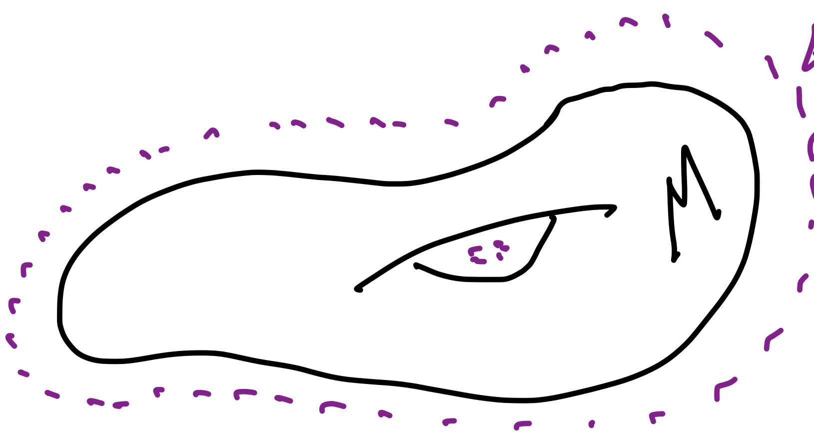
<u>Ex</u>: A bordism between S^1 and $S^1 \sqcup S^1$:

Stable Normal Bundle

- given an embedding of a manifold

$$M \hookrightarrow \mathbb{R}^k$$

it has a non-unique normal bundle



normal bundle cuts out a slightly bigger copy of M from $\mathbb{R}^k$

HOWEVER:

- if we let k get large enough, this bundle becomes unique up to isotopy

- we call this the stable normal bundle of M

- this is a real vector bundle whose structure group is

$$\mathcal{O} := \varinjlim_{n} (O(n))$$

infinite orthogonal matrices

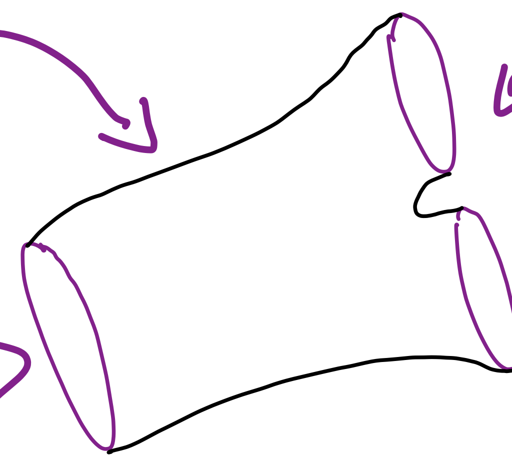
- given a group map $G \to O$, we say M has a G-structure if we can reduce the structure group from O to G

$$\begin{tikzcd} & BG \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "N"'] & BO \end{tikzcd}$$

G-Bordism

- we can now ask that our manifolds M, N, P all have G-structure and that the G-structure on P restricts to that of M and N.

now they're G-manifolds!



The Ring Ω^G

- now let $\Omega^G_n = \left\{ \begin{array}{c} \text{n-dim'l manifolds} \\ \text{w/ G structure} \end{array} \right\} \Big/ \sim$

where $M \sim N$ if there is a bordism between M and N

Remarkable Fact:

The sets $\{\Omega^G_n\}$ form a graded ring, with $\amalg$ as addition and $\times$ as multiplication!

Thom, '54

Ex:

Dold manifolds, '56

$$\Omega^{O}_{*} \cong \mathbb{Z}/2[x_i \mid i \geq 0,\ i \neq 2^k - 1]$$

Novikov, '60

$$\Omega^{U}_{*} \cong \mathbb{Z}[x_i \mid i \geq 0,\ i = 2k]$$

Wall, '60

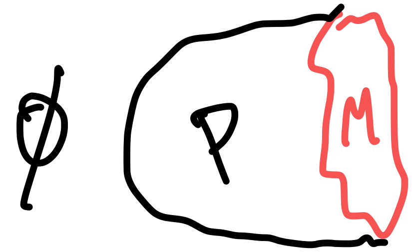
Ω^{SO}_{*} is complicated but known

and $\Omega^{SO}_{*} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^{2i} \mid i \geq 0]$

Ω^{Spin}_{*} - don't know ring struct.!

Ω^{String}_{*} - not fully known!

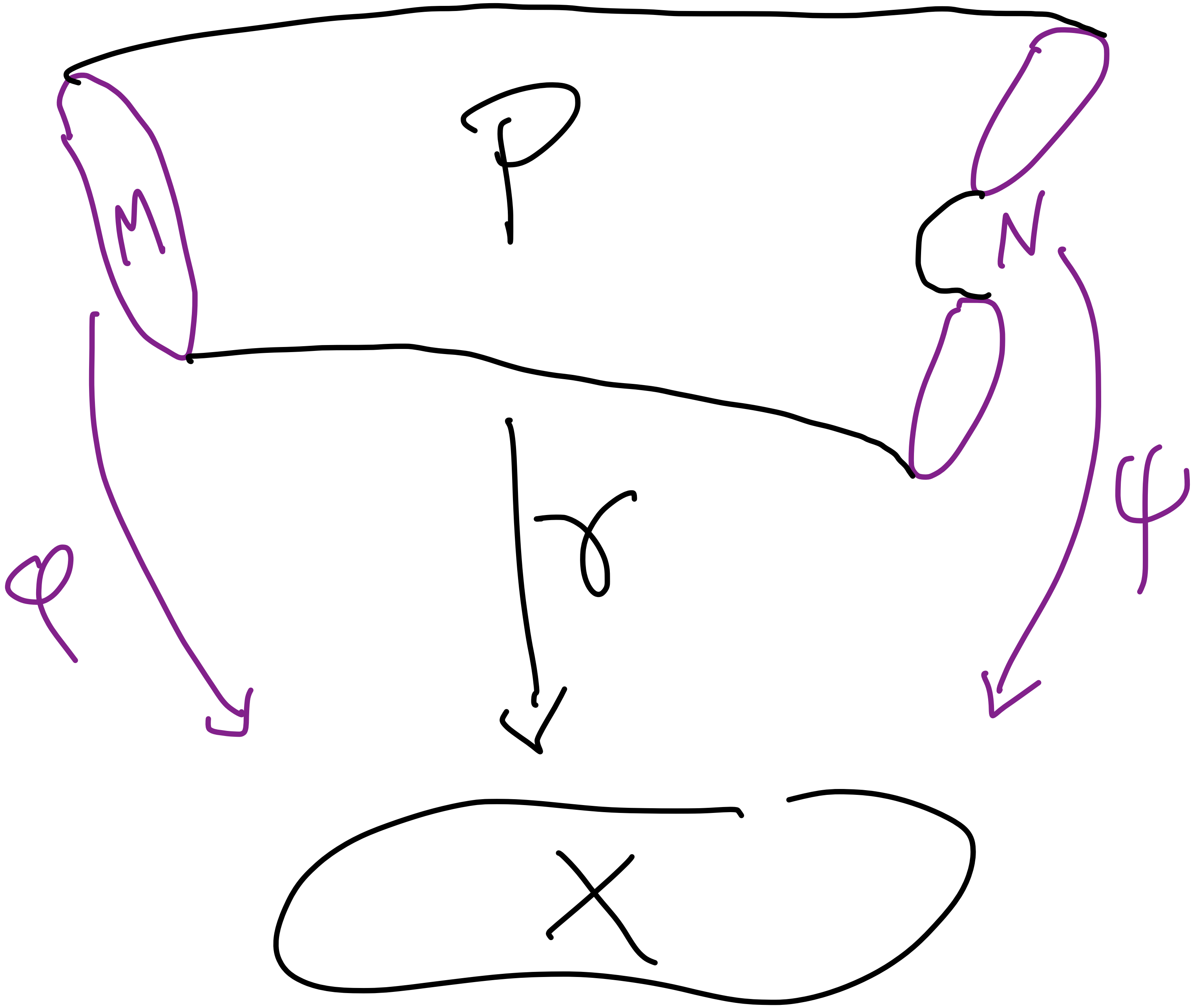
Bordism Homology Theories



- more generally, for a space X, define

$$\Omega_n^G(X) = \left\{ \begin{array}{l} \text{n-manifolds with} \\ \text{G-structure over } X \end{array} \right\} \Big/ \sim$$

where $(M \xrightarrow{\varphi} X) \sim (N \xrightarrow{\psi} X)$ if there is a bordism between M and N over X, $(P \xrightarrow{\gamma} X)$, which restricts to φ and ψ



(φ, ψ, γ are just cts. maps)

Thm (Atiyah, '60): The sets $\Omega^G_*(X)$ assemble into a generalized homology theory.

- in other words, a well-behaved machine

$$\mathrm{Top} \xrightarrow{\Omega^G_*} \mathrm{GrAb}$$

graded abelian groups

- in particular

$$\Omega_*^G(pt.) \cong \Omega_*^G$$

<u>Rmk:</u> Given a map of structure groups $H \to G$, we get a natural transformation $\Omega_*^H(-) \longrightarrow \Omega_*^G(-)$ ("inducing up" the structure group)

A Very Special Case

Q'n: What if $G = \{e\}$?

- structure group is trivial, so all manifolds must be framed or parallelized.

$$\underbrace{\Omega_*^{fr}(X) \cong \pi_*^s(X)}_{\text{Thom, '54 Pontrjagin, '59}}$$

the (stable) homotopy groups of X

Def'n: For a pointed space X,

$$\pi_k^s(X) = \lim_{\overrightarrow{n}} \underbrace{\pi_{n+k}(S^n \wedge X)}$$

Theorem (Freudenthal, '38): This sequence becomes constant for large n.

- $\pi_k^s(X)$ approximates $\pi_k(X)$ and is often easier to compute

Note: For any structure group G there is a map

$$\Omega_*^{fr}(X) \longrightarrow \Omega_*^{G}(X)$$

(we can always think of e as an element of G)

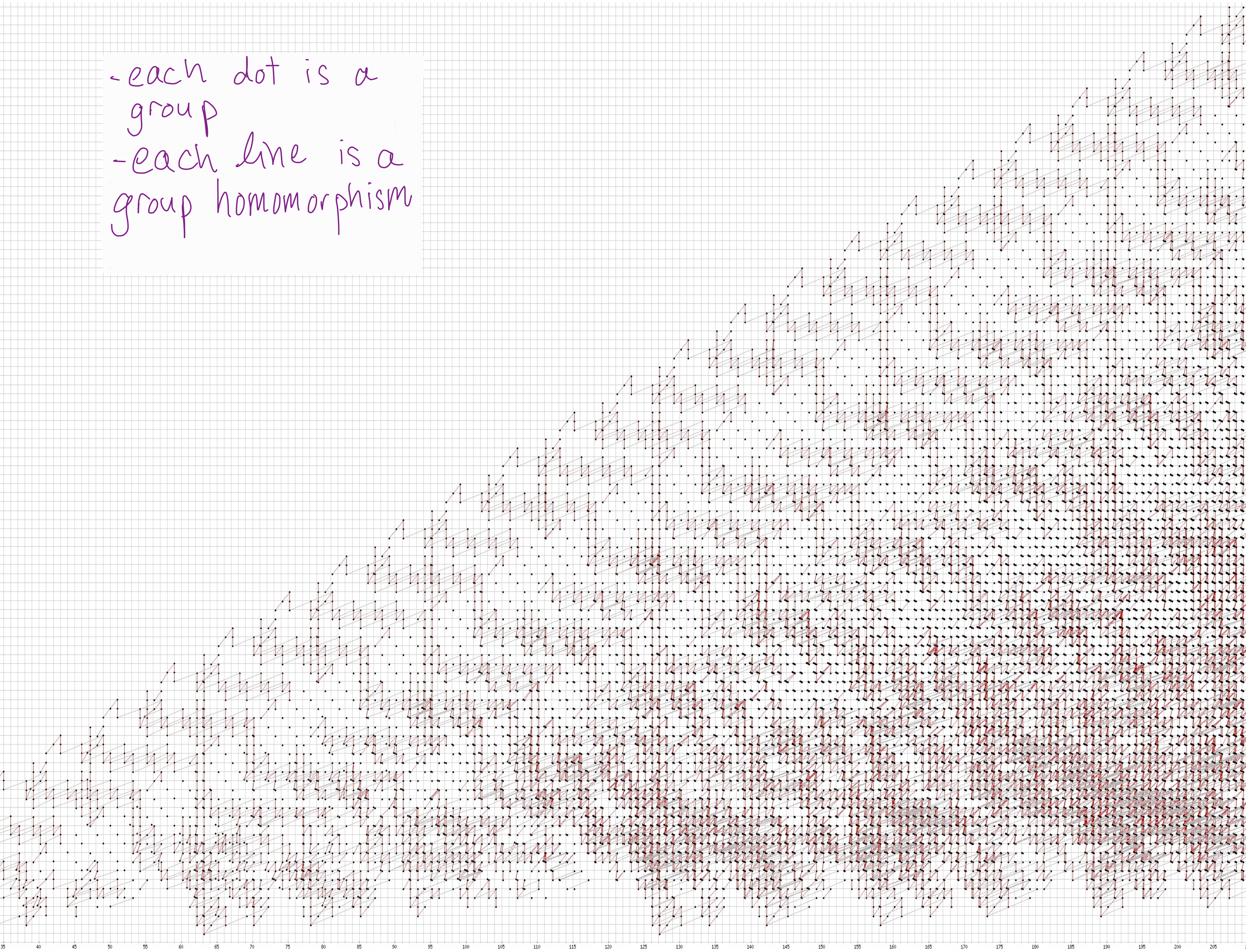
Classically:

There is a spectral sequence for "descending" $\Omega^G_*(X)$ to obtain $\Omega^{fr}_*(X) \cong \pi^S_*(X)$.

Novikov, '67
Adams, '71

It's a rather complicated spectral sequence

- each dot is a group
- each line is a group homomorphism



Goal: Get a more conceptual understanding of the relationship between π_*^s and Ω_*^G.

Waldhausen, '78: Do algebra with the invariants of algebraic topology directly, rather than their outputs

Idea: These maps assemble into a Galois extension of invariants, i.e. the Galois structure themselves exists on the functors

$$\Omega_*^{fr}(-) \longrightarrow \Omega_*^{G}(-)$$

Rmk: There is a tensor product on homology theories with respect to which our functors Ω_*^G are all ring (monoid) objects!

Generalized Galois Extensions

Recall:

A map of fields $\varphi: L \to K$ is Galois, with Galois group G iff G acts on K and $L \cong K^G$ where $K^G = \{k \in K \mid gk = k \ \forall g \in G\}$.

One reason this is useful:

Galois Descent: We can learn about L-modules by studying K-modules and then taking G-fixed points

$$\mathrm{Mod}_K \xrightarrow{\text{descent}} \mathrm{Mod}_L$$

(e.g. computing Picard groups)

- Now let $L[G]^* = \mathrm{Hom}_L(L[G], L)$ be the dual of the group ring of G.

<u>Prop</u>: $\varphi : L \longrightarrow K$ is G-Galois iff
(Exercise!)
(i) $L[G]^*$ coacts on K:

$$\rho : K \longrightarrow K \otimes_L L[G]^*$$

(ii) K is an $L[G]^*$ "torsor":

$$K \otimes_L K \cong K \otimes_L L[G]^*$$

(iii) L is the $L[G]^*$ cofixed points of K:

$$L \cong K^{\mathrm{co}L[G]^*} = \{k \in K \mid \sigma(k) = k \otimes 1\}$$

More generally:

Def'n: Given a map of rings $R \xrightarrow{\phi} S$ and a Hopf-algebra H, ϕ is H-Galois if:

(i) H coacts on S over R

(ii) $S \otimes_R S \cong S \otimes H$

(iii) $\underbrace{R \cong S^{coH} = \{ s \in S \mid s \longmapsto s \otimes 1 \}}_{\text{allows for "Galois descent"}}$

Still have Galois descent:

$$\mathrm{Mod}_S \rightsquigarrow \mathrm{Mod}_R$$

taking H-cofixed points

- Galois cohomology, in terms of the group G, is replaced by a kind of Hopf-algebra cohomology

Thm (B.): The natural transformation

$$\Omega^{fr}_*(-) \longrightarrow \Omega^G_*(-)$$

is H-Galois for $H = BG$

Rmk: BG is not a Hopf-algebra!
However, it is a bimonoid object of Top_*:

diagonal $\rightarrow$ $\Delta: BG \longrightarrow BG \wedge BG$

$\mu: BG \wedge BG \longrightarrow BG$

inherited from G (in many cases)

What does this mean for a fixed space X?

- coaction:

$$\Omega_*^G(X) \longrightarrow \Omega_*^G(BG \wedge X)$$

(Thom diagonal)

- torsor:

$$(\Omega_*^G \otimes \Omega_*^G)(X) \cong \Omega_*^G(BG \wedge X)$$

tensor product of invariants

(Thom isomorphism)

- descent:

$\pi_*^s(X)$ can be obtained as the derived cofixed points of $\Omega_*^G(X)$ w/r/t BG coaction

i.e. running the spectral sequence is computing (extremely complicated) Galois cohomology

Rmk: BG is very much not the dual group ring of any kind of group, which is why we need to generalize Galois extensions.

Fundamental Theorem

More generally:

<u>Thm (B.)</u>: Given a fiber sequence of structure groups $H \to G \to G/H$, there is an intermediate extension:

$$\overbrace{\underbrace{\Omega^{fr}_*(-) \longrightarrow \Omega^H_*}_{BH}(-) \longrightarrow \underbrace{\Omega^G_*(-)}_{B(G/H)}}^{BG}$$

Ex: Whitehead tower of O:

$$
\begin{tikzcd}
String \arrow[d] & \\
Spin \arrow[r] \arrow[d] & K(\mathbb{Z},3) \\
SO \arrow[r] \arrow[d] & K(\mathbb{Z}/2,1) \\
O \arrow[r] & K(\mathbb{Z}/2,0)
\end{tikzcd}
$$

$$\underbrace{\Omega^{fr}_* \to \Omega^{String}_*}_{BString} \underbrace{\to \Omega^{Spin}_*}_{K(\mathbb{Z},4)}$$

$$\underbrace{\Omega^{fr}_* \to \Omega^{Spin}_*}_{BSpin} \underbrace{\to \Omega^{SO}_*}_{K(\mathbb{Z}/2,2)}$$

$$\underbrace{\Omega^{fr}_* \to \Omega^{SO}_*}_{BSO} \underbrace{\to \Omega^{O}_*}_{K(\mathbb{Z}/2,1)}$$