A mysterious appearance of quantum probability in projective geometry

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These are notes for a talk I gave at Caltech on May 14th, 2025 at the Information, Geometry and Physics Seminar organized by Matilde Marcolli and Yassine El Maazouz. These are based on the papers [BN24] and [Bea25].

1 Γ-sets and Hyperoperations

Definition 1.1. A Γ-set is a pointed functor $X\colon \mathcal{F}in_* \to \mathcal{S}et_*$ from the category of finite pointed sets to the category of all pointed sets. From here on out I'll take the skeleton of $\mathcal{F}in_*$ spanned by the pointed set $\langle n\rangle = \{*, 1, 2, \dots, n\}$ and all pointed functions between them.

Connes and Consani use Γ-sets as a model for modules over the field with one element [CC16; CC20; CC19]. I won't lean too heavily on that in this talk, but it is relevant since one way to interpret this work is that the $\mathbb{F}_1$-module associated to the trivial projective geometry on a single point is the functor of Dynkin systems on finite (pointed) sets (a.k.a. "set representable orthomodular posets"). This is all part of one approach to making sense of Tits' intuition about the field with one element in combinatorial settings (as opposed to arithmetic geometry a la Kapranov and Smirnov).

Definition 1.2. We name some useful functions in $\mathcal{F}in_*$.

1. Let $\alpha_i\colon \langle n\rangle \to \langle n-1\rangle$ denote the function which is the identity on all $j \leq i$ and takes j to $j-1$ whenever $j > i$. In other words, it "merges" i and $i+1$.
2. Let $\pi_i\colon \langle n\rangle \to \langle 1\rangle$ be the function that takes i to 1 and everything else to $*$. In other words, π_i is the function that "projects out" the i^{th} coordinate.
3. Write $\varepsilon\colon \langle 0\rangle \to \langle 1\rangle$ for the inclusion of the base point and write $\eta\colon \langle 1\rangle \to \langle 0\rangle$ for the terminal morphism.

Remark 1.3. Note that by taking coproducts of the above functions one can describe a copy of Δ^{op} inside of $\mathcal{F}in_*$. For instance, the maps α_i become the inner face maps. This is equivalently the image of the simplicial circle $\Delta^1/\partial\Delta^1\colon \Delta^{\mathrm{op}} \to \mathcal{F}in_* \subset \mathcal{S}et_*$.

Example 1.4. Here's a well-known example of a Γ-set. Let $(M, +, 0)$ be the data of a commutative monoid (in $\mathcal{S}et$). Then there is a Γ-set HM which has: $HM_n = M^n$; $HM(\alpha_1)\colon M^2 \to M$ the operation $+$ of M; $HM(\pi_i)$ the projections $M^n \to M$; and $HM(\varepsilon)$ the inclusion of 0. You can probably figure out what the rest of the functor looks like on your own.

Let's take a moment to see how functors out of $\mathcal{F}in_*$ might be good at modeling commutative, unital, binary "operations." Consider the following diagram in $\mathcal{S}et_*$, for X a Γ-set.

$$\begin{tikzcd}
X_2 \arrow[r, "X(\pi_1)\times X(\pi_2)"] \arrow[d, "X(\alpha_1)"] & X_1 \times X_1 \\
X_1 \arrow[d, "X(\varepsilon)"] & \\
X_0 \arrow[u, bend left, "X(\eta)"] &
\end{tikzcd}$$

If we wanted to take two elements of X_1, say $(x, y) \in X_1 \times X_1$ and combine them according to X, we could do the following: take their inverse image under $X(\pi_1) \times X(\pi_2)$ to get a (possibly empty) subset of X_2 and then apply $X(\alpha_1)$ to get a subset of X_1. This defines a function $X_1 \times X_1 \to \mathcal{P}(X_1)$ which we think of as a binary *hyperoperation* on X_1. Because X is a functor and $\alpha_1 \circ \tau = \alpha_1$ in $\mathcal{F}in_*$ (where τ is the twist map on $\langle 2 \rangle$), this "operation" is necessarily commutative. Of course, when $X = HM$ the map is $X(\pi_1) \times X(\pi_2)$ is the identity, and this recovers the original binary operation of M. The map $X(\varepsilon)$ recovers the identity element, and functoriality makes it all "work."

Definition 1.5. Given a Γ-set $X\colon \mathcal{F}in_* \to \mathcal{S}et_*$, write $\boxplus_X\colon X_1 \times X_1 \to \mathcal{P}(X_1)$ for the associated hyperoperation.

Remark 1.6. Note that if $X(\pi_1) \times X(\pi_2)$ is *injective* then the operation $\boxplus_X$ is singly-valued but not necessarily defined on all of X_1. If, on the other hand, the map is *surjective*, then the operation is defined everywhere but potentially multi-valued.

Definition 1.7. Given a Γ-set X, write $\Sigma_n^X\colon X_n \to X_1 \times \cdots \times X_1$ for the product morphism $\pi_1 \times \pi_2 \times \cdots \times \pi_n$. Say that X is *special* if Σ_n^X is an isomorphism for all n. Note that the case of $n = 0$ is satisfied for all of our Γ-sets since we've assumed pointedness.

Proposition 1.8. *The construction $M \mapsto HM$ extends to a fully faithful functor $H\colon \mathrm{CMon} \hookrightarrow \Gamma\mathcal{S}et_*$ whose essential image is the full subcategory of special Γ-sets.*

The functor H is often called the Eilenberg-MacLane functor. We can embed $\Gamma\mathcal{S}et_*$ into $\Gamma\mathcal{T}op_*$ and Segal showed that the latter can be used to model connective spectra (i.e. cohomology theories with no negative degrees). The image of H is then precisely the Eilenberg-MacLane spectra.

In this talk, however, the ability of Γ-sets to model hyperoperations is the focus. If we enforced "specialness" we would lose all the examples of a more combinatorial nature.

2 Plasmas and Projective Geometries

Definition 2.1. Define a *plasma* to be a set X with a distinguished *identity element* $0 \in X$ and a hyperoperation $\star\colon X \times X \to \mathcal{P}(X)$ satisfying the following conditions:

1. For all $x, y \in X$, $x \star y = y \star x$.
2. For $x \in X$, $x \in x \star 0$.

A define a *morphism of plasmas* $(X, \star, 0) \to (Y, \boxplus, e)$ to be a function $f\colon X \to Y$ such that $f(x \star y) \subseteq f(x) \boxplus f(y)$. It's straightforward to check that this gives a category which we denote Plas.

Definition 2.2. Let $(X, \star, 0)$ and $(Y, \boxplus, e)$ be plasmas. Then we define their coproduct $(X \vee Y, \heartsuit, [0])$ to be the pointed set $(X, 0) \vee (Y, e)$ with the operation

$$x \heartsuit y = \begin{cases} x \star y & x, y \in X \\ x \boxplus y & x, y \in Y \\ \varnothing & \text{else} \end{cases}$$

Clearly the shared basepoint $[0]$ remains a weak identity.

It's not hard to check that the above construction makes $X \vee Y$ in the categorical coproduct of Plas.

Proposition 2.3. *If X is a Γ-set then the hyperoperation $\boxplus_X\colon X_1 \times X_1 \to \mathcal{P}(X_1)$, along with the identity element $X(\varepsilon)(*) \in X_1$, makes X_1 into a plasma. In fact, this defines an essentially surjective functor $\Gamma\mathcal{S}et_* \to$ Plas.*

Example 2.4. Any Abelian (hyper)group or commutative (hyper)monoid of course has an underlying plasma. An important example for us comes from taking the Krasner hyperfield $\mathbb{K} = \{0, 1\}$ whose hyperaddition is given by the rules $0 + 0 = \{0\}$, $1 + 0 = 0 + 1 = \{1\}$ and $1 + 1 = \{0, 1\}$. The multiplicative structure of $\mathbb{K}$ will not be relevant in this talk.

Example 2.5. It one takes the inclusion functor $\mathcal{F}in_* \hookrightarrow \mathcal{S}et_*$, which Connes and Consani call $\mathfrak{s}$ but think of as $\mathbb{F}_1$, the associated plasma structure on $\mathbb{F}_1(\langle 1 \rangle) = \{0, 1\}$, has 0 as a strict identity element and $1 + 1 = \varnothing$.

Example 2.6. Let S be a set. Then the powerset $\mathcal{P}(S)$ has a plasma structure according to the rule:

$$A \uplus B = \begin{cases} A \cup B & \text{if } A \cap B = \varnothing \\ \varnothing & \text{if } A \cap B \neq \varnothing \end{cases}$$

The following is not terribly hard to check.

Theorem 2.7. *Let $\mathcal{P}(n)$ denote the power set of the set $\{1, 2, \ldots, n\}$. Then by taking $\langle n \rangle$ to $\mathcal{P}(n)$ and a map $\phi\colon \langle n \rangle \to \langle m \rangle$ to the function $\phi^{-1}\colon \mathcal{P}(m) \to \mathcal{P}(n)$, we obtain a functor $\mathcal{P}\colon \mathcal{F}in_*^{\mathrm{op}} \to$ Plas.*

Corollary 2.8. *There is a fully faithful right adjoint* $\widehat{H}\colon \mathrm{Plas} \to \Gamma\mathcal{S}et_*$ *to the left Kan extension depicted below which is corepresented by the functor* $\mathcal{P}$.

$$\begin{tikzcd} \mathcal{F}in_*^{\mathrm{op}} \arrow[r, "\mathcal{P}"] \arrow[d, "よ"'] & \mathrm{Plas} \\ \Gamma\mathcal{S}et_* \arrow[ur, dashed, "\mathrm{Lan}_{よ}\mathcal{P}"'] & \end{tikzcd}$$

You should compare this to the left Kan extension defining geometric realization for simplicial sets. In that case, you've got a functor $\Delta \to \mathcal{T}op$ that takes $[n]$ to the geometric n-simplex. You then left Kan extend this functor along the Yoneda embedding $よ\colon \Delta \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}et) = s\mathcal{S}et$. The resulting functor $s\mathcal{S}et \to \mathcal{T}op$ takes each simplex of the simplicial set to a geometric simplex and then glues them together according to the face and degeneracy maps. The right adjoint to this is given by mapping out of the cosimplicial object which in level n is the geometric n-simplex.

In the case of the above theorem, we have a functor which takes each $\langle n\rangle$ to the plasma $\mathcal{P}(n)$, which we think of as the "basic building block" of the functor we're building. By then left Kan extending we're saying that for a Γ-set X then: first you should install a copy of $\mathcal{P}(n)$ for each "n-simplex" (which in this case is a morphism out of a representable, $よ\langle n\rangle \to X$); next you should glue these copies of $\mathcal{P}(n)$ together along the morphisms of $\mathcal{F}in_*$. The right adjoint $\widehat{H}$ being corepresented by $\mathcal{P}$ then means, among other things, that $\widehat{H}A_n = \mathrm{Plas}(\mathcal{P}(n), A)$. This is comparable to computing the singular simplicial set of a space X by assembling the mapping sets $\mathcal{T}op(\Delta^n, X)$.

This can be summarized by saying that the adjunction $\mathrm{Lan}_{よ}\mathcal{P}\colon \Gamma\mathcal{S}et_* \rightleftarrows \mathrm{Plas}\colon \widehat{H}$ is a kind of "nerve-realization" adjunction. An important point to make here, however, is that the "realization" functor in this setting is a kind of truncation functor. It is very close to the functor that takes a functor $\mathcal{F}in_* \to \mathcal{S}et_*$ and restricts it to a functor on the full subcategory spanned $\langle 0\rangle$, $\langle 1\rangle$ and $\langle 2\rangle$. This is a kind of 2-coskeletonization adjunction.

In any case, this means that $\widehat{H}A_n$ is relatively easy to compute for any fixed n. One simply needs to compute the set of plasma morphisms $\mathcal{P}(n) \to A$. We will leverage that to understand what $\widehat{H}$ on a particular example of interest.

Another kind of object important to this talk is a *projective geometry*. There are many definitions of projective geometries, some of which are equivalent, and others of which define more or less general classes of object. We will use one of the three definitions given in [FF00].

Definition 2.9. A *projective geometry* is a set G equipped with a ternary "collinearity" relation $\ell \subseteq G \times G \times G$ satisfying the following conditions (where a, b, c, d, p, q are points of G):

1. For every a and b, $(a, b, a) \in \ell$.
2. If $p \neq q$ then $(a, p, q) \in \ell$ and $(b, p, q) \in \ell$ imply that $(a, b, p) \in \ell$.

3. If (p,a,b) and (p,c,d) are in ℓ then there is some q such that $(q,a,c) \in \ell$ and $(q,a,d) \in \ell$.

A morphism of projective geometries is a function $G \to H$ preserving collinearity. Write Proj for the category of projective geometries.

Remark 2.10. For the purposes of [BN24], a projective geometry was defined as a pointed, simple, finitary matroid satisfying the projective law. This is equivalent to the above definition but requires significantly more axioms to describe.

Theorem 2.11 ([NR23]). *There is a fully faithful inclusion*

$$\Phi\colon \mathrm{Proj} \hookrightarrow \mathrm{Plas}.$$

In the actual paper, Reyes and Nakamura embed projective geometries into a category of algebraic structures that they call *mosaics*. However, it's pretty clear that mosaics are a full subcategory of plasmas, so everything checks out.

The intuition behind that inclusion is roughly the following: given a projective geometry G, define a closure operator $\mathcal{P}(G) \to \mathcal{P}(G)$ which takes a subset $A \subseteq G$ to the smallest subspace containing A (where "subspace" means a collection of points closed under collinearity); this defines a matroid which faithfully encodes the geometry; since morphisms between projective geometries are typically only partially defined functions, and we want actual functions, we append a disjoint basepoint and then replace the closure operator with a hyperoperation encoding the same data. When we start with a projective geometry, this construction is fully faithful. If we were to start in the middle, with simple pointed matroids, it would only be faithful.

Corollary 2.12. *The composite of $\widehat{H}$ and Φ gives a fully faithful embedding*

$$\Theta\colon \mathrm{Proj} \to \Gamma\mathcal{S}et_*.$$

A natural question then (at least for me) is to try to compute Φ and Θ for some examples. To begin, I considered the simplest possible geometry: that with one point and no lines. The next simplest, for me at least, are the "trivial" geometries of [FF00].

Definition 2.13. Let G_n denote the projective geometry with n elements and the ternary relation $\ell \subseteq G_n^3$ defined by saying $(a,b,c) \in \ell$ if and only if $|\{a,b,c\}| \leq 2$. In other words, there is a unique line through every pair of points and no other lines.

Remark 2.14. As closure spaces, the geometries G_n correspond to the closure operators $id\colon \mathcal{P}(n) \to \mathcal{P}(n)$. In both cases it's straightforward to check that their automorphism groups are the symmetric groups. In other words, these are the canonical "thin" geometries of Tits.

Theorem 2.15. *The plasma $\Phi(G_n)$ is canonically isomorphic to the n-fold coproduct of $\mathbb{K}$ with itself. In particular, Φ of the one point geometry is isomorphic to $\mathbb{K}$.*

Recall that Φ is fully faithful. Therefore the group of automorphisms of $\Phi(G_n)$ should be Σ_n. It's not hard to check that, indeed, $\mathrm{Aut}_{\mathrm{Plas}}(\vee_n\mathbb{K}) \cong \Sigma_n$. Also note that $\Phi(G_n)$ is *not* $\vee_n\mathbb{F}_1$. One could argue that Tits' "field with one element" really was $\mathbb{K}$, and that $\mathbb{F}_1$ as Connes and Consani have defined it is somehow something deeper.

3 Dynkin Systems

Extending the computation along $\widehat{H}$ will involve Dynkin systems so let's get that out of the way. A good reference for these is [DW23], although they were first described by Dynkin in [Dyn61].

Definition 3.1. Let X be a set and $\mathcal{P}(X)$ its power set. We say that a family $\mathcal{A} \subseteq \mathcal{P}(X)$ is a ***pre-Dynkin system*** if it satisfies the following:

1. $\varnothing \in \mathcal{A}$.
2. If $U \in \mathcal{A}$ then $X - U \in \mathcal{A}$.
3. If $U, V \in \mathcal{A}$ and $U \cap V = \varnothing$ then $U \cup V \in \mathcal{A}$.

If the third condition above can be extended to arbitrary countable collections of mutually disjoint sets, we say that $\mathcal{A}$ is a Dynkin system (without the "pre-").

Dynkin systems seem to show up in probabilistic situations when it's not possible to know whether or not things are happening "at the same time." One might have well defined probabilities for certain events in the system, but not know how to compute the probability that they *both happen.* An important result about Dynkin systems is the so-called $\pi\lambda$-theorem which says that if a Dynkin system is closed under intersections then it is also closed under unions (hence a σ-algebra).

These things have also shown up in so-called "quantum logic." A Dynkin system on X is the same thing as what people call an *orthomodular poset* in $\mathcal{P}(X)$ (or a "set-representable" orthomodular poset, since not all orthomodular posets can be embedded in power set lattices). These generalize the orthomodular lattices that show up in quantum mechanics as lattices of closed subspaces of Hilbert spaces, which go back to Birkhoff and Von Neumann [BN36]. See also [Sup66; Gud69; Gud84]. It's not clear to me how relevant this kind of thing is in actually thinking about quantum mechanical systems. According to Wikipedia at least, "modern philosophers reject quantum logic as a basis for reasoning...".

In any case, it's not hard to see that Dynkin systems assemble into a functor out of $\mathcal{F}in_*$. Write $\mathtt{Dynk}_n$ for the set of Dynkin systems on the set $\langle n \rangle$.

Lemma 3.2. *If $\phi \colon \langle n \rangle \to \langle m \rangle$ is a function of pointed sets and $\mathcal{A} \subseteq \mathcal{P}(\langle n \rangle)$ is a Dynkin system on $\langle n \rangle$, the family*

$$\phi_* \mathcal{A} = \{U \subseteq \langle m \rangle : \phi^{-1}(U) \in \mathcal{A}\}$$

is a Dynkin system on $\langle m \rangle$. The assignments $\langle n \rangle \mapsto \mathtt{Dynk}_n$ and $(\phi \colon \langle n \rangle \to \langle m \rangle) \mapsto (\phi_* \colon \mathtt{Dynk}_n \to \mathtt{Dynk}_m)$ define a functor $\mathtt{Dynk} \colon \mathcal{F}in_* \to \mathcal{S}et_*$.

Remark 3.3. Note that the functor $\mathtt{Dynk}$ doesn't really care about the sets being pointed. We could define it identically on $\mathcal{F}in$. However, the fact that its domain is $\mathcal{F}in_*$ restricts the ways in which we can pass a Dynkin system from one set to another.

The primary theorem of this work, which I can prove but still don't really understand, is the following.

Theorem 3.4. *There is a natural isomorphism of* Γ*-sets*

$$\Phi(G_n) \cong \widehat{H}(\vee_n \mathbb{K}) \xrightarrow{\cong} \vee_n \mathtt{Dynk}$$

where the right-hand wedge sum denotes (object-wise) coproduct in $\mathrm{Fun}_*(\mathcal{F}in_*, \mathcal{S}et_*)$.

In other words, the $\mathbb{F}_1$-module associated to the one-point trivial projective geometry is the functor of Dynkin systems.

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