Topological Hochschild homology of $X(n)$

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Theorem 1. *The spectrum $X(n)$, which is the Thom spectrum of the inclusion $\Omega SU(n) \hookrightarrow \Omega SU \simeq BU \to BGL_1(\mathbb{S})$, is of characteristic η. In particular $X(2)$ is the versal $\mathbb{E}_1$-$\mathbb{S}$-algebra of characteristic η (as described in Definition 4.3 of [ACB14]).*

Proof. We use [ACB14] in a crucial way. Recall that $X(2)$ is the Thom spectrum of the inclusion $i : \Omega S^3 \simeq \Omega SU(2) \hookrightarrow BU \to BGL_1(\mathbb{S})$. Note that this morphism, as the inclusion of a subset of based loops on a group, ΩSU, is a two fold loop map, and as such a morphism of $\mathbb{E}_2$-algebras in $\mathcal{T}$. Let $\tilde{i}$ be the induced $\mathbb{E}_1$-morphism. We have the following equivalences of mapping spaces:

$$map_{\mathbb{E}_1-alg}(\Omega\Sigma S^2, BGL_1(\mathbb{S})) \simeq map_{\mathcal{T}}(S^2, BGL_1(\mathbb{S}) \simeq map_{\mathcal{T}}(S^1, GL_1(\mathbb{S})).$$

Since $\pi_1(GL_1(\mathbb{S})) \cong \pi_1(\mathbb{S}) \cong \mathbb{Z}/2$ we have that this map is either null homotopic or unique up to homotopy. In other words, $\tilde{i}$ is homotopic to η, the generator of $\pi_1(\mathbb{S})$. Indeed, the preceding sequence of equivalences implies, by Theorem 4.10 of [ACB14], that $X(2) \simeq \mathbb{S}/\!\!/_{\mathbb{E}_1}\eta$, the versal $\mathbb{E}_1$-algebra over $\mathbb{S}$ of characteristic η. Moreover, as $X(n)$ admits a morphism of $\mathbb{E}_1$-ring spectra (actually of $\mathbb{E}_2$-ring spectra) $X(2) \to X(n)$ for every n, we have that the $X(n)$ must be an $\mathbb{E}_1$-$\mathbb{S}$-algebra of characteristic η. In particular, the composition $\Sigma\mathbb{S} \xrightarrow{\eta} \mathbb{S} \to X(n)$ is nullhomotopic. □

Remark 1. Of course it's not necessary to use the machinery of characteristics of structured ring spectra to notice that η is trivial in $X(n)_*$, but the identification of $X(2)$ as the versal $\mathbb{E}_1$-$\mathbb{S}$-algebra of characteristic η seems interesting in its own right.

Theorem 2. *The topological Hochschild homology of $X(n)$ as an $\mathbb{E}_2$-ring spectrum, denoted here by $THH(X(n))$, is equivalent to $X(n) \wedge SU(n)_+$.*

Proof. Here we use [BCS10] is a crucial way. In particular, we recall Theorem 2 of that paper which gives $THH(X(n)) = X(n) \wedge M(\eta \circ Bi)$, where $\eta \circ Bi$ here refers to the morphism

$$B\Omega SU(n) \simeq SU(n) \xrightarrow{Bi} B^2GL_1(\mathbb{S}) \xrightarrow{\eta} BGL_1(\mathbb{S}).$$

Since $X(n)$ is of characteristic η, we have that the composition $B^2GL_1(\mathbb{S}) \xrightarrow{\eta} BGL_1(\mathbb{S}) \to BGL_1(X(n))$ is nullhomotopic, where $BGL_1(\mathbb{S}) \to BGL_1(X(n))$ is just $BGL_1(-)$ of the unit map of $X(n)$. This implies that $M(\eta \circ Bi)$ is $X(n)$-oriented, so by the associated Thom isomorphism we have $X(n) \wedge M(\eta \circ Bi) \simeq X(n) \wedge SU(n)_+$. □

Conjecture 1. *Recall that the morphism of $\mathbb{E}_2$-ring spectra $X(n) \to X(n+1)$ is a Hopf-Galois extension with associated spectral Hopf-algebra ΩS^{2n+1}, though of as the base space of the fibration $\Omega SU(n) \to \Omega SU(n+1) \to \Omega S^{2n+1}$. Then the above results, as well as the results of [BCS10] suggest that one might have relative THH spectra:*

$$THH_{X(n)}(X(n+1)) \simeq X(n+1) \wedge S^{2n+1}_+.$$

The first theorem above also suggests that there may be a morphism $\overline{\eta} : \Omega S^{2n+1} \to BGL_1(X(n))$ whose Thom spectrum is $X(n+1)$ as an $X(n)$-algebra.

References

[ACB14] Omar Antolín-Camarena and Tobias Barthel, *A simple universal property of Thom ring spectra*, 2014, arXiv:1411.7988.

[BCS10] Andrew J. Blumberg, Ralph L. Cohen, and Christian Schlichtkrull, *Topological Hochschild homology of Thom spectra and the free loop space*, Geom. Topol. **14** (2010), no. 2, 1165–1242.