Picard Spaces and Orientations

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Picard Spaces

This talk will be about the following structures, most of which are described in [ABG+14, ABG18]:

1. In general, $(\mathcal{C}, \otimes, \mathbb{I})$ will be a symmetric monoidal ∞-category. Often I'll want to talk about an $\mathbb{E}_\infty$-ring spectrum R and $\mathcal{C}$ will be the ∞-category of left R-modules, LMod_R.

2. I'll write $\mathrm{Pic}(\mathcal{C})$ for the maximal ∞-groupoid of $\mathcal{C}$ on the $\otimes$-invertible objects of $\mathcal{C}$. Note that $\mathrm{Pic}(\mathcal{C})$ is an ∞-groupoid and inherits a symmetric monoidal structure from $\mathcal{C}$; in other words, it's an infinite loop space. Moreover there is a symmetric monoidal, faithful inclusion $\mathrm{Pic}(\mathcal{C}) \hookrightarrow \mathcal{C}$.

3. As constructed, $\mathrm{Pic}(\mathcal{C})$ has a canonical basepoint, namely $\mathbb{I}$ itself, and I'll write $\mathrm{BGL}_1(\mathbb{I})$ for the connected component of this point. I'll also write $\mathrm{GL}_1(\mathbb{I})$ for $\Omega\,\mathrm{BGL}_1(\mathbb{I})$. This notation will make sense in a second.

4. All of these spaces are infinite loop spaces so they have corresponding spectra $\mathfrak{pic}(\mathcal{C})$, $\mathfrak{bgl}_1(\mathbb{I})$ and $\mathfrak{gl}_1(\mathbb{I})$.

Proposition 1 ([ABG+14]). *There is an equivalence*

$$\Omega\,\mathrm{Pic}(\mathcal{C}) \simeq \Omega\,\mathrm{BGL}_1(\mathcal{C}) \simeq \mathrm{GL}_1(\mathbb{I})$$

where $\mathrm{GL}_1(\mathbb{I}) = Aut_{\mathcal{C}}(\mathbb{I})$.

I'll be focusing on a very special case of this construction:

Example 2. Let R be an $\mathbb{E}_\infty$-ring spectrum and let $\mathcal{C} = \mathrm{LMod}_R$ be its ∞-category of left modules. In this case, I'll write $\mathrm{Pic}(R)$ instead of $\mathrm{Pic}(\mathrm{LMod}_R)$, along with $\mathrm{BGL}_1(R)$ and $\mathrm{GL}_1(R)$.

In this case, we have a really nice construction of the space of units:

Proposition 3 ([ABG+14]). *There is a pullback of ∞-groupoids*

$$\begin{array}{ccc} \mathrm{GL}_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \hookrightarrow & \pi_0(R) \end{array}$$

in which the right vertical arrow is projection onto connected components (or 0-truncation of the space).

Corollary 4. *For a commutative ring spectrum R, $\pi_0(\mathrm{GL}_1(R)) \cong \pi_0(R)^\times$ and $\pi_i(\mathrm{GL}_1(R)) \cong \pi_i(R)$ for all $i > 0$.*

Note also that if R is non-connective (i.e. it has non-trivial negative degree homotopy groups) then $\mathrm{GL}_1(R)$ and $\mathrm{GL}_1(R^{\geq 0})$ are equivalent, where $R^{\geq 0}$ is the connective cover of R. This of course implies that $\mathrm{BGL}_1(R) \simeq \mathrm{BGL}_1(R^{\geq 0})$. However, in general, $\mathrm{Pic}(R)$ and $\mathrm{Pic}(R^{\geq 0})$ need not be equivalent.

Lemma 5. *If R is a commutative ring spectrum then $\Sigma^i R$ is invertible for all $i \in \mathbb{Z}$.*

The proof of this is to notice that

$$\begin{aligned}\Sigma^i R \otimes_R \Sigma^{-i} R &\simeq \mathbb{S}^i \otimes R \otimes_R \mathbb{S}^{-i} \otimes R \\ &\simeq \mathbb{S}^i \otimes \mathbb{S}^{-i} \otimes R \\ &\simeq R\end{aligned}$$

Example 6. A good example of how Pic can be different a ring spectrum and its connective cover comes from the complex K-theory spectrum KU and connective K-theory ku. Recall that KU is 2-periodic so $\Sigma^i KU \simeq \Sigma^{i+2} KU$ for all $i \in \mathbb{Z}$. This means that $\mathrm{Pic}(KU)$ has a 2-torsion element, namely ΣKU, since $\Sigma KU \otimes_{KU} \Sigma KU \simeq \Sigma^2 KU \simeq KU$.

On the other hand, ku is not periodic, so it's got an infinite cyclic subgroup generated by Σku. This doesn't necessarily mean that $\mathrm{Pic}(KU)$ and $\mathrm{Pic}(ku)$ can't be equivalent (maybe they're both $\mathbb{Z}/2 \oplus \mathbb{Z}$). In this case however there's a computation to be done, which shows that, indeed, $\pi_0(\mathrm{Pic}(KU)) \cong \mathbb{Z}/2$ and $\pi_0(\mathrm{Pic}(ku)) \cong \mathbb{Z}$. This is given as an example in [MS16].

Example 7. There are lots of other computations of Picard groups (by which I mean π_0 of Picard spaces) that have been done:

- $\pi_0(\mathrm{Pic}(\mathbb{S})) \cong \mathbb{Z}$,
- $\pi_0(\mathrm{Pic}(KO)) \cong \mathbb{Z}/8$
- $\pi_0(\mathrm{Pic}(TMF)) \cong \mathbb{Z}/576$,
- $\pi_0(\mathrm{Pic}(Tmf)) \cong \mathbb{Z} \oplus \mathbb{Z}/24$.

An area of active interest is trying to determine $\mathrm{Pic}(L_n\mathbb{S})$, the chromatic localizations of the sphere spectrum, for various primes. Some of these are known but to my knowledge there is still quite a bit to do here.

R-line Bundles and Thom Spectra

Definition 8. Let X be an ∞-groupoid and R a commutative ring spectrum. Then an *R-line bundle* on X is a map of ∞-groupoids $f\colon X \to \mathrm{Pic}(R)$. Given such a map, we write Mf for the colimit of the composite $X \to \mathrm{Pic}(R) \hookrightarrow \mathrm{LMod}_R$ and call Mf the *Thom spectrum* of f. Note that Mf is in fact an R-module, not just a spectrum.

Proposition 9. *The above construction can be lifted to a colimit preserving functor* $M\colon \mathcal{S}_{/\,\mathrm{Pic}(R)} \to \mathrm{LMod}_R$.

Recall that $\mathcal{S}_{/\,\mathrm{Pic}(R)}$ is equivalent, by straightening, to the ∞-category $\mathrm{Fun}(\mathrm{Pic}(R), \mathcal{S})$. This makes it clear that there are two symmetric monoidal structures on it. These are the pointwise symmetric monoidal structure which tensors two functors together object by object in $\mathcal{S}$ and the Day convolution monoidal structure, which is a bit more complicated. These monoidal structures both have nice descriptions in $\mathcal{S}_{/\,\mathrm{Pic}(R)}$ however. Given two maps $X \to \mathrm{Pic}(R)$ and $Y \to \mathrm{Pic}(R)$, their pointwise tensor product is given by the pullback

$$\begin{array}{ccc} X \times_{\mathrm{Pic}(R)} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Pic}(R) \end{array}$$

and the Day convolution monoidal structure corresponds to the composite

$$X \times Y \to \mathrm{Pic}(R) \times \mathrm{Pic}(R) \to \mathrm{Pic}(R)$$

in which the final map is the group structure of $\mathrm{Pic}(R)$. We can say precisely what $\mathbb{E}_k$-algebras are in both of these categories.

Lemma 10. *The $\mathbb{E}_k$-algebra objects of $\mathcal{S}_{/\operatorname{Pic}(R)}$ with respect to the pointwise monoidal structure are those which factor through $\operatorname{Alg}_{\mathbb{E}_k}(\operatorname{LMod}_R)$. With respect to the Day convolution monoidal structure, they are the $\mathbb{E}_k$-monoidal maps $X \to \operatorname{Pic}(R)$.*

The Thom spectrum functor preserves colimits but it does not generally preserve limits, so it's not monoidal with respect to the pointwise monoidal structure, however:

Theorem 11 ([ACB19])**.** *The functor $M\colon \mathcal{S}_{/\operatorname{Pic}(R)} \to \operatorname{LMod}_R$ is symmetric monoidal with respect to the Day convolution monoidal structure on $\mathcal{S}_{/\operatorname{Pic}(R)}$.*

Corollary 12 ([Lew78])**.** *If $f\colon X \to \operatorname{Pic}(R)$ is a map of $\mathbb{E}_k$-monoidal ∞-groupoids then Mf is an $\mathbb{E}_k$-R-algebra.*

We have a good supply of interesting Thom spectra arising from cobordism spectra.

Example 13. Recall the so-called J-homomorphism $J\colon \mathrm{O} \to \mathrm{GL}_1(\mathbb{S})$ giving an action of the stable orthogonal group on the sphere spectrum. This admits a delooping, $B\mathrm{O} \to \mathrm{BGL}_1(\mathbb{S}) \hookrightarrow \operatorname{Pic}(\mathbb{S})$. We can then compose this with $\operatorname{Pic}(\mathbb{S}) \to \mathrm{Sp}$ to get a Thom spectrum. It is a result of [ABG+14], relying on the results of [LMSM86], that the resulting spectrum is indeed MO, the unoriented cobordism spectrum. For any of the classical maps $B\mathrm{U} \to B\mathrm{O}$, $B\mathrm{SO} \to B\mathrm{O}$, etc., we obtain the usual cobordism spectra, MU, MSO, etc.

Orientations

A useful fact to record here is the following:

Lemma 14. *Let X be an ∞-groupoid, R a commutative ring spectrum, and A an R-algebra, and $f\colon X \to \operatorname{Pic}(R)$ a map of ∞-groupoids.*

1. *If f is nullhomotopic then $Mf \simeq R \otimes \Sigma^\infty_+ X$.*
2. *If $- \otimes_R A\colon \operatorname{Pic}(R) \to \operatorname{Pic}(A)$ denotes the map which is the restriction of the usual basechange functor then $M(f \otimes_R A) \simeq Mf \otimes_R A$.*

The two facts above aren't too hard to prove if one is familiar with the relevant technology. The first one is practically the definition of the way in which LMod_R is tensored over $\mathcal{S}$, and the second is essentially noticing that $- \otimes_R A$ commutes with colimits.

Definition 15. Given a map of ∞-groupoids $f\colon X \to \operatorname{Pic}(R)$ and an R-algebra A, we say that Mf is A-oriented if the composite

$$X \to \operatorname{Pic}(R) \to \operatorname{Pic}(A)$$

is nullhomotopic.

Proposition 16. *The space of A-orientations of a Thom spectrum Mf is equivalent to the space of R-module maps $Mf \to A$.*

This immediately tells us that whenever $f\colon X \to \operatorname{Pic}(R)$ is A-oriented there's an equivalence $Mf \otimes_R A \simeq A \otimes \Sigma^\infty X$. But we can do a little bit better and actually identify the map that carries that isomorphism (this is worked out in detail in [Bea]). First we'll need an alternative description of orientations:

Proposition 17. *The space of A-orientation of Mf is equivalent to the space of R-module maps $u\colon Mf \to A$ with the following property:*

> *Let Mx denote the Thom spectrum associated to the composition with an inclusion of a point $x \to X \to \operatorname{Pic}(R)$ and let u_x denote the composite $Mx \to Mf \to A$ (where Mx maps to Mf by functoriality of the Thom spectrum functor). Note that $Mx \simeq R$. Then the composite*
>
> $$A \simeq Mx \otimes_R A \xrightarrow{u_x \otimes A} A \otimes_R A \xrightarrow{\mu_A} A$$
>
> *is an equivalence, where μ_A is the multiplication map of A.*

This, combined with a little extra algebraic structure that we can put on Mf, lets us give a concrete description of the Thom isomorphism.

Theorem 18. *Let $f\colon X \to \mathrm{Pic}(R)$ be a map of ∞-groupoids and A be an R-algebra.*

1. *If f is nullhomotopic then $Mf \simeq R \otimes \Sigma^\infty_+ X$ is a cocommutative coalgebra in LMod_R. If X is an $\mathbb{E}_k$-space and f is $\mathbb{E}_k$-monoidal then $R \otimes \Sigma^\infty_+ X$ is a cocommutative bialgebra whose algebra structure is $\mathbb{E}_k$-monoidal.*
2. *For any f the Thom spectrum M is an $R \otimes \Sigma^\infty_+ X$-comodule with structure map $\Delta_f\colon Mf \to \Sigma^\infty_+ X \otimes Mf$.*
3. *If Mf is A-oriented by an R-module map $u\colon Mf \to A$ then the composite*

$$Mf \otimes_R A \xrightarrow{\Delta_f \otimes A} \Sigma^\infty_+ X \otimes Mf \otimes_R A \xrightarrow{X \otimes u \otimes A} \Sigma^\infty_+ X \otimes A \otimes_R A \xrightarrow{X \otimes \mu_A} \Sigma^\infty_+ X \otimes A$$

is an equivalence, called the Thom isomorphism. If $f\colon X \to \mathrm{Pic}(R)$ is an $\mathbb{E}_k$-map and $u\colon Mf \to A$ is a map of R-algebras then the Thom isomorphism is a map of R-algebras.

A special case of the above is when Mf is an algebra map. Then the identity map $Mf \to Mf$ satisfies the conditions of being an Mf-orientation. This gives us classical equivalences like, for instance, $MU \otimes MU \simeq MU \otimes \Sigma^\infty_+ B\mathrm{U}$.

A Classical Example

It's natural to ask why precisely I'm calling such things "orientations." The basic idea is that it's a way to trivialize the R-line bundle by basechanging. This, to me, is a little bit confusing though, since, for instance, oriented vector bundles aren't necessarily trivial. The correct thing to say is that oriented vector bundles are $H\mathbb{Z}$-oriented, which is what I now want to explain.

First, recall that $\pi_0(\mathrm{Pic}(\mathbb{S})) \cong \mathbb{Z}$, so we have that $\mathrm{Pic}(\mathbb{S}) \simeq \mathbb{Z} \times \mathrm{BGL}_1(\mathbb{S})$. Now suppose I've got a real, finite dimensional vector bundle $\xi\colon X \to B\mathrm{O}(n)$. This determines a map $\tilde{\xi}\colon X \to \mathbb{Z} \times B\mathrm{O}$ into the n component which, by composition with the J-homomorphism, determines a map $X \to \mathrm{Pic}(\mathbb{S})$, again into the n-component. Note that this actually defines an $\mathbb{S}^n$-bundle on X, since the n-component of $\mathrm{Pic}(\mathbb{S})$ corresponds to the invertible $\mathbb{S}$-module $\mathbb{S}^n$. Now convince yourself that the colimit of this functor is $\Sigma^{\infty+n}_+ X^\xi$, the suspension spectrum of the Thom space of ξ. For instance, you can try to think about taking the associated sphere bundle on X, taking fiberwise suspension spectra, and then taking the colimit.

To keep things simple, I'm going to desuspend this vector bundle by subtracting the trivial bundle from it n times. This means that I get a map $X \to B\mathrm{O}$ instead of $\{n\} \times B\mathrm{O}$. It also means that the resulting Thom spectrum is just $\Sigma^\infty_+ X^\xi$, instead of its n-fold suspension. These two constructions only differ by a degree shift, so I'm not losing any information.

Now further suppose that ξ is oriented, i.e. there is a lift

$$\begin{array}{ccc} & & B\mathrm{SO} \\ & \nearrow & \downarrow \\ X & \xrightarrow{\xi} & B\mathrm{O} \end{array}$$

along the simply connected cover of $B\mathrm{O}$. Because I'm landing in $B\mathrm{O}$, I now have a composite $X \to B\mathrm{SO} \to B\mathrm{O} \to \mathrm{BGL}_1(\mathbb{S})$. It's a fact that $\mathrm{GL}_1(H\mathbb{Z}) \simeq \mathbb{Z}^\times \simeq \mathbb{Z}/2$, so $\mathrm{BGL}_1(H\mathbb{Z}) \simeq K(\mathbb{Z}/2, 1)$. So tensoring with $H\mathbb{Z}$ gives a composite

$$X \to B\mathrm{O} \to B\mathrm{SO} \to BGL_1(\mathbb{S}) \to \mathrm{BGL}_1(H\mathbb{Z}) \simeq K(\mathbb{Z}/2, 1)$$

Finally, recall that $K(\mathbb{Z}/2, 1)$ represents $H^1(-; \mathbb{Z}/2)$ and that, for instance by the Hurewicz theorem, $H^1(B\mathrm{SO}; \mathbb{Z}/2) \cong 0$. In other words, there are no non-nullhomotopic maps $B\mathrm{SO} \to K(\mathbb{Z}/2, 1)$. Putting this all together tells me that the Thom spectrum of the above composite (in $\mathrm{LMod}_{H\mathbb{Z}}$) is $\Sigma^\infty_+ X^\xi \otimes H\mathbb{Z} \simeq \Sigma^\infty_+ X \otimes H\mathbb{Z}$. Taking homotopy groups then gives the classical Thom isomorphism

$$H_*(X^\xi; \mathbb{Z}) \cong H_*(X; \mathbb{Z})$$

If you go look up "Thom isomorphism" on Wikipedia, you'll see that there's a degree shift in the above isomorphism. I got rid of this when I shifted my vector bundle from $\{n\} \times BO$ to $\{0\} \times BO$, but that wasn't strictly necessary.

Note that you could do the same thing with $\mathrm{BGL}_1(H\mathbb{Z}/2) \cong B(*) \simeq *$ (the ring $\mathbb{Z}/2$ doesn't have very many units!). In this case, *every* map $X \to \mathrm{BGL}_1(H\mathbb{Z}/2)$ becomes contractible. It follows that every vector bundle is oriented by $\mathbb{Z}/2$-homology.

Torsor Structure

One of my favorite things about Thom spectra is that they have this structure that noncommutative geometers call a "Hopf-Galois extension," which really just means that if you take *Spec* then you get a torsor (cf. [Rog08]). Note that if you have a G-Galois extension of fields (or rings) then taking *Spec* gives a G-torsor over *Spec* of the base field. This is a partial answer to why these things are called Hopf-Galois extensions. Here's the definition of torsor in the simplest case:

Definition 19. Let $X \to Y$ be a map of sets and G be a group acting on X over Y, i.e. such that the following diagram commutes

$$\begin{array}{ccc} X \times G & \longrightarrow & X \\ & \searrow & \downarrow \\ & & Y \end{array} .$$

We say that X is a G-torsor over Y if the map

$$X \times G \xrightarrow{\Delta_X \times G} X \times_Y X \times G \to X \times_Y X$$

is an isomorphism, where the last map is the action of G on X.

If you let Y be the one element set in the above definition then you get precisely that G acts freely and transitively on X. If you replace X and Y with spaces then you'll get precisely the definition of a principal G-bundle on Y. You can also rewrite the definition above for schemes, and you'll get the notion of a G-torsor in algebraic geometry.

Now note that for a Thom spectrum Mf associated to an $\mathbb{E}_\infty$-map $f\colon X \to \mathrm{Pic}(R)$ (hence a commutative R-algebra) we have the two following data:

1. A coaction
$$Mf \xrightarrow{\Delta_f} \Sigma^\infty_+ X \otimes Mf$$
2. An equivalence
$$Mf \otimes_R Mf \xrightarrow{\Delta_f \otimes Mf} \Sigma^\infty_+ X \otimes Mf \otimes_R Mf \xrightarrow{X \otimes \mu_{Mf}} \Sigma^\infty_+ X \otimes Mf$$

Recall also that $\Sigma^\infty_+ X \otimes R$ is a (co)commutative R-bialgebra, so $Spec(\Sigma^\infty_+ X \otimes R)$ is an affine (spectral) Abelian group scheme over $Spec(R)$. So Thom spectra have this very natural "algebro-geometric" structure. Note that when $X \to \mathrm{Pic}(R)$ is only $\mathbb{E}_k$-monoidal, you can't sensibly take *Spec* because you're not dealing with commutative rings (you can take the opposite category but then tensor products don't turn into Cartesian products, so things behave very strangely). However, you can still write down the above data and try to do *noncommutative* geometry.

References

[ABG18] Matthew Ando, Andrew J. Blumberg, and David Gepner, *Parametrized spectra, multiplicative Thom spectra and the twisted Umkehr map*, Geom. Topol. **22** (2018), no. 7, 3761–3825. MR3890766

[ABG+14] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, and Charles Rezk, *An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology*, J. Topol. **7** (2014), no. 3, 869–893.

[ACB19] Omar Antolín-Camarena and Tobias Barthel, *A simple universal property of Thom ring spectra*, J. Topol. **12** (2019), no. 1, 56–78. MR3875978

[Bea] Jonathan Beardsley, *On bialgebras, comodules, descent data and Thom spectra in ∞-categories.* Forthcoming in Homotopy, Homology and Applications.

[Lew78] L.G. Lewis, *The stable category and generalized Thom spectra*, Ph.D. Thesis, 1978. The University of Chicago.

[LMSM86] L. G. Lewis Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure. MR866482

[MS16] Akhil Mathew and Vesna Stojanoska, *The Picard group of topological modular forms via descent theory*, Geom. Topol. **20** (2016), no. 6, 3133–3217. MR3590352

[Rog08] John Rognes, *Galois extensions of structured ring spectra/Stably dualizable groups*, Mem. Amer. Math. Soc. **192** (2008), no. 898, viii+137. MR2387923