Some notes on the category of p-local harmonic spectra

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July 22, 2013

Introduction

In [9], Mark Hovey and Neil Strickland determine that the lattice of localizing subcategories of the $E(n)$-local stable homotopy category is isomorphic to the Boolean algebra of subsets of the set with n elements. They also show that the lattice of localizing subcategories of the $K(n)$-local stable homotopy category isomorphic to the Boolean algebra $\mathbb{F}_2$. Since $E(n)$ is Bousfield equivalent to the wedge of the first n Morava K-theories, it seems natural to the author to ask if this result can be extended to the infinite wedge of Morava K-theories, or even arbitrary wedges. Indeed, it is shown that for a wedge of Morava K-theories indexed by some $I \subseteq \mathbb{N}$, the Bousfield lattice of the stable homotopy category localized at that wedge is precisely the Boolean algebra of subsets of I. Note that this result only applies to Bousfield lattices, but can probably be extended to all localizing subcategories.

1 The Harmonic Lattice

Definition 1. We work in the stable homotopy category localized at a fixed prime $p \in \mathbb{N}$. We introduce the following notation to improve readability:

(i) For $I \subseteq \mathbb{N}$, let $K(I) = \vee_{i \in I} K(i)$, where $K(i)$ is the ith Morava K-theory. For instance, the harmonic spectrum $\bigvee_{n \in \mathbb{N}} K(n)$ will be denoted by $K(\mathbb{N})$.

(ii) The localization functor associated to $K(I)$ will be denoted by L_I, except when $I = \{0, 1, \ldots, n\}$, in which case we will write L_n for the associated localization functor, as is traditional.

(iii) By $\mathcal{H}$ we mean the stable homotopy category localized at $K(\mathbb{N})$, i.e. the p-local harmonic category.

(iv) For an arbitrary spectrum X, let $\mathrm{supp}(X) = \{n : K(n) \wedge X \not\simeq *\}$ and $\mathrm{cosupp}(X) = \{n : K(n) \wedge X \simeq *\}$.

(v) Let $\langle X\rangle_{\mathcal{H}}$ denote the collection of $Y \in \mathcal{H}$ such that $X \wedge Y \simeq *$, i.e. the "harmonic" Bousfield class of X.

Remark. From [8], Theorem 3.5.1, we know that $\mathcal{H}$ has a natural structure satisfying the following:

(i) $\mathcal{H}$ is a triangulated category with a closed symmetric monoidal structure compatible with the triangulation.

(ii) $\mathcal{H}$ has a set $\mathcal{G}$ of strongly dualizable objects such that the smallest localizing subcategory of $\mathcal{H}$ containing $\mathcal{G}$ is $\mathcal{H}$ itself.

(iii) $\mathcal{H}$ has all coproducts.

(iv) Every cohomology functor is representable on $\mathcal{H}$.

We also know from [10] Theorem 3.1 (assuming Vopěnka's principle, but most likely under weaker hypotheses) that the Bousfield classes of $\mathcal{H}$ form a set rather than a proper class, hence the notion of Bousfield lattice is well-defined in this context (i.e. we're allowed to quantify over arbitrary subsets of the lattice to forms meets and joins, so on and so forth). It follows from [8], Theorem 3.7.3 that $\langle K(n)\rangle_{\mathcal{H}}$ is minimal in the Bousfield lattice of $\mathcal{H}$ for every $0 \leq n < \infty$.

Lemma 1. *If X is harmonic then $K(n) \wedge X \simeq *$ for every n if any only if $X \simeq *$.*

Proof. It is obvious that if $X \simeq *$ then $K(n) \wedge X \simeq *$ for every n. Suppose then that $K(n) \wedge X \simeq *$ for every n and X is harmonic. Then $K(\mathbb{N}) \wedge X \simeq *$ so X is $K(\mathbb{N})$-equivalent to $*$, but X is $K(\mathbb{N})$-local, hence $X \simeq *$. □

Proposition 1. *If X and Y are objects of $\mathcal{H}$, then $X \wedge Y \simeq *$ if and only if $Y \wedge K(n) \simeq *$ for every $n \in \operatorname{supp}(X)$.*

Proof. Suppose first that $Y \wedge X \simeq *$, i.e. $Y \in \langle X\rangle_{\mathcal{H}}$. We note that, since $X \wedge K(n) \simeq \vee_d \Sigma^d K(n)$ for every $n \in \operatorname{supp}(X)$, $\langle X\rangle_{\mathcal{H}} \curlywedge \langle K(n)\rangle_{\mathcal{H}} \leq \langle K(n)\rangle_{\mathcal{H}}$. But since $\langle K(n)\rangle_{\mathcal{H}}$ is minimal and $\langle X \wedge K(n)\rangle_{\mathcal{H}} \neq 0$, we know that $\langle X\rangle_{\mathcal{H}} \curlywedge \langle K(n)\rangle_{\mathcal{H}} = \langle K(n)\rangle_{\mathcal{H}}$. But for an arbitrary lattice, if $a \curlywedge b = b$ then $b \leq a$. Hence $\langle X\rangle_{\mathcal{H}} \geq \langle K(n)\rangle_{\mathcal{H}}$ for every $n \in \operatorname{supp}(X)$. In other words, the collection of X-acyclics is contained in the collection of $K(n)$-acyclics. Hence, $Y \wedge K(n) \simeq *$ for every $n \in \operatorname{supp}(X)$.

Now suppose that $Y \wedge K(n) \simeq *$ for every $n \in \operatorname{supp}(X)$. Then we know that $(X \wedge Y) \wedge K(n) \simeq *$ for every $n \in \operatorname{supp}(X)$ and for every $n \in \operatorname{cosupp}(X)$, that is, for every $n \in \mathbb{N}$. However, $X \wedge Y$ is harmonic, so, because it is $K(\mathbb{N})$-local but annihilated by every $K(n)$, $X \wedge Y$ is contractible. □

Corollary 1. *The Bousfield lattice of $\mathcal{H}$ (which we will denote by $\mathcal{BH}$) is the lattice generated by the collection of Morava K-theories.*

Proof. The proposition above shows that for X an object of $\mathcal{H}$, X is $\mathcal{H}$-Bousfield equivalent to the wedge of Morava K-theories which support it. Additionally, every Morava K-theory is harmonic, as well as the sphere, which is supported by all the Morava K-theories. Hence the Bousfield lattice of $\mathcal{H}$ is in fact a complete, atomic Boolean algebra on the Morava K-theories. It is isomorphic to the Boolean algebra of subsets of $\mathbb{N}$. □

Corollary 2. *For $K(I)$ and $K(J)$ in $\mathcal{BH}$, the meet $\langle K(I)\rangle \curlywedge \langle K(J)\rangle$ is equal to $\langle K(I) \wedge K(J)\rangle = \langle K(I \cap J)\rangle$.*

Remark. The above proposition and corollaries extend Hovey and Strickland's proofs of the structure of the $E(n)$-local and $K(n)$-local Bousfield lattices. It is clear that it can be generalized to the stable homotopy category localized at an arbitrary wedge of Morava K-theories. That is to say, the Bousfield lattice of the stable homotopy category localized at $K(I)$ for some $I \subseteq \mathbb{N}$ is precisely the Boolean algebra of subsets of I.

It also follows from the above statements that the telescope conjecture holds in the harmonic category. However, as all harmonic spectra are BP-local, and the telescope conjecture holds BP-locally [6], we know that already.

References

[1] P. Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew Math. **588** (2005) 149-168.

[2] A.K. Bousfield, *The Boolean algebra of spectra*, Comment. Math Helv. **54** (1979) 368-377.

[3] A.K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257-281.

[4] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. **142** (1969) 43-60.

[5] M. Hopkins and D. Ravenel, *Suspension spectra are harmonic*, Boletin Soc. Mat. Mexicana (1992) 271-280.

[6] M. Hovey, *Bousfield localization functors and Hopkins' chromatic splitting conjecture*, Contemp. Math. **181** (1995) 225-250.

[7] M. Hovey and J. Palmieri, *The Structure of the Bousfield Lattice*, Contemp. Math. **239** (1999) 175-197.

[8] M. Hovey, J. Palmieri and N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128(610)** (1997).

[9] M. Hovey and N. P. Strickland, *Morava K-theories and localization*, Mem. Amer. Math. Soc. **139(666)** (1999).

[10] S. Iyengar and H. Krause, *The Bousfield Lattice of a Triangulated Category and Stratification*, 2011, arxiv:math/1105.1799v1.

[11] P. Johnstone, Stone Spaces, Cambridge Univ. Press (1982).

[12] A. Neeman: *Triangulated Categories*, Princeton Univ. Press (2001).

[13] T. Ohkawa. *The injective hull of homotopy types with respect to generalized homology functors*, Hiroshima Math. J. **19** (1989), 631-639.

[14] D. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106(2)** (1984) 351-414.