Group Theory for ∞-Groups

UIUC Topology Seminar

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1 Background

This talk represents joint work with Landon Fox. This work builds on concepts introduced by K. Hess, E. Farjoun and especially M. Prasma.

Definition 1. An ∞-group is a grouplike $\mathbb{E}_1$-monoid in $\mathcal{S}$. Write $\mathcal{G}\mathrm{rp}$ for the ∞-category of ∞-groups.

Definition 2. Let X be a space and G an ∞-group. A G action on X is a functor $\rho\colon BG \to \mathcal{S}$ with $\rho(*) \simeq X$. We define the quotient space X/G to be $colim(\rho)$.

Theorem 3 (May, Stasheff)**.** *The functor $\Omega\colon \mathcal{S}_*^{\geq 1} \to \mathcal{G}\mathrm{rp}$ is an equivalence whose inverse is the delooping functor B.*

Lemma 4. *Let G be an ∞-group. Then G acts canonically on itself by a functor $can_G\colon BG \to \mathcal{S}$ whose associated cocartesian fibration is $* \simeq EG \to BG$.*

Remark 5. Note that a G-action on X corresponding to a functor $\rho\colon BG \to \mathcal{S}$ is the same data, by the Grothendieck construction, as a Kan fibration $E \to BG$ with fiber equivalent to X.

Proposition 6. *Let $\rho\colon BG \to \mathcal{S}$ be a G-action on a space X with associated fiber sequence $X \to E \xrightarrow{\pi} BG$. Then $X/G = colim(\rho) \simeq E$.*

Proof. Note that the fibration $\pi\colon E \to B$ associated to the action corresponds to the pullback of the universal fibration:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"] & U_{\mathcal{S}} \arrow[d] \\ BG \arrow[r, "\rho"] & \mathcal{S} \end{tikzcd}$$

In general, given a cocartesian fibration $F \to Z$ with corresponding functor $Z \to \mathcal{C}at_\infty$, the latter has colimit equivalent to F with its cocartesian morphisms inverted. However, in this case, because $E \to BG$ is a Kan fibration (of ∞-groupoids), every morphism in E is already invertible, so $colim(\rho) \simeq E$. □

Corollary 7. *Let $\phi\colon H \to G$ be a morphism of ∞-groups and $can_G \circ B\phi\colon BH \to BG \to \mathcal{S}$ be the associated action of H on G. Then there is an equivalence $G/H \simeq fib(B\phi)$ and a fiber sequence $H \to G \to G/H \to BG \to BG$.*

2 Normality Data

Definition 8. Let $\phi\colon H \to G$ be a morphism of ∞-groups. A *normality datum* for ϕ is a morphism $\nu\colon BG \to X$, with X a pointed connected space, whose fiber $F \to BG$ is equivalent to $B\phi\colon BH \to BG$. In light of the preceding proposition we will write G/H for ΩX whenever ν is a normality datum for $G \to H$.

Remark 9. It is possible for a map $\phi\colon H \to G$ to have more than one normality datum. Note that normality data for the terminal ∞-group map $G \to *$ are in bijection with $\mathbb{E}_2$-structures on G. Therefore the map $G \to *$ can have more than one normality datum. There are more complicated examples as well.

Remark 10. A normal map $H \to G$ is the same data as a local system on BG whose fiber is a loop space.

Definition 11. Let $\phi\colon X \to Y$ be a morphism of spaces. Then we write its unique epi-mono factorization as $X \to im(\phi) \to Y$. Recall that $X \to im(\phi)$ is a surjection on π_0 and $im(\phi) \to Y$ is the inclusion of the connected components of Y in the image of $\pi_0\phi$.

Theorem 12 (First Isomorphism Theorem)**.** *Let $\phi\colon H \to G$ be a morphism of ∞-groups. Then*

1. *There is a group map $im(\phi) \to G$.*
2. *The fiber $fib(\phi)\colon F \to H$ admits a normality datum such that $im(\phi) \simeq H/F$.*

Proof. For the first statement, note that we can construct $im(\phi)$ as the pullback below:

$$\begin{tikzcd} im(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d] \\ \pi_0(H) \arrow[r] & \pi_0(G) \end{tikzcd}$$

This is a pullback in $\mathcal{G}\mathrm{rp}$ and agrees with the pullback of underlying spaces. This gives a unique factorization

$$\begin{tikzcd} H \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & im(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d] \\ & \pi_0(H) \arrow[r] & \pi_0(G) \end{tikzcd}$$

For the second statement, consider the factorization of ∞-group maps $H \to im(\phi) \hookrightarrow G$. Because ϕ is a group map, $im(\phi)$ must contain the base point, i.e. the group identity, of G. It follows that $H \to G$ and $H \to im(G)$ have equivalent fibers. Therefore there is a fiber sequence of ∞-groups $F \to H \to im(\phi)$. Delooping gives $BH \to Bim(\phi)$, which is the desired normality datum for $fib(\phi)$. □

Remark 13. There is a natural interpretation of the Second Isomorphism Theorem, but it is false. It may be the case that there is a more clever restatement of it for ∞-groups that is true. However, recall that the Second Isomorphism Theorem *does not* hold for topological groups.

Remark 14. The natural way to state the Third Isomorphism Theorem would be as follows: suppose that we have maps of ∞-groups $\phi\colon K \to G$ and $\psi\colon H \to G$ with normality data ν and η respectively such that ϕ factors through ψ via some $\theta\colon K \to H$. Then:

1. There is a canonical normality datum for the composite $\theta\colon K \to H$.
2. There is a canonical normality datum for $H/K \to G/K$ such that $(G/K)/(H/K) \simeq G/H$ is an equivalence of ∞-groups.

There is clearly an action of K on G and if we *assume* that K is normal in H then one can show that there is also an action of H/K on G/K. Using this structure there is an equivalence $(G/K)/(H/K) \simeq G/H$ of spaces. However, it is not clear that, a priori, there should be any normality datum for the map of ∞-groups $H/K \to G/K$, which would be necessary for asking whether or not that equivalence lifts to one of ∞-groups.

3 Orbits and Stabilizers

Remark 15. Let $F\colon BG \to \mathcal{S}$ be a G-space, hence $F(*) = X$ is "the space" on which G is acting. The corresponding Kan fibration is then the fiber sequence $X \to E \to BG$, for some space E. By Proposition 6 we know that $E \simeq X/G$.

The action of G on X, from this point of view, is the monodromy action of $\Omega BG \simeq G$ on the fiber. In particular, if we choose a 0-simplex $x \in X$ and demand that $X \to E$ be pointed (thereby determining a base point in E), we can take another fiber to obtain a fiber sequence $G \to X \to E$. This should be thought of as the principal G bundle coming from pulling back the universal G-bundle along $E \to BG$:

$$\begin{tikzcd} G \arrow[r, "\simeq"] \arrow[d] & G \arrow[d] \\ X \arrow[r] \arrow[d] & EG \arrow[d] \\ E \arrow[r] & BG \end{tikzcd}$$

Note that the construction in Remark 15 relied on a choice of base point $x \in X$.

Definition 16. Let $x \in X$ be a 0-simplex in a G-space X, and let $G \to X \to E$ be the resulting fiber sequence. The we write Orb_x for the image of the map $G \to X$ so constructed.

The intuition behind calling this space the "orbits" of $x \in X$ is as follows. A choice of $x \in X$ induces a choice of base point $e \in E$. But $E = X/G$, so the fiber of $X \to E$ over e is all points of X which can be translated to x via the G-action. Indeed, the map $G \to X$ constructed by taking the fiber is, at the level of 0-simplices, the map that takes $g \in G$ to $gx \in X$. Therefore the image of $G \to X$ is the connected components of X which are "hit" by translating x.

Remark 17. Taking "images" in higher mathematics feels unnatural. For instance, if X is connected then $Orb_x = X$ for any point $x \in X$. It may be that there is a subtler notion of "orbits" which is more satisfying. For the time being however, we are able to show that our chosen space of orbits is equivalent to a reasonable construction of a "stabilizer group."

Definition 18. Let $X \to E \to BG$ be a G-action on X and choose a zero simplex $x \in X$ determining a 0-vertex $e \in E$. Then we define the stabilizer group $Stab_x$ to be $\Omega_e E = \Omega_e X/G$.

Proposition 19 (Orbit-Stabilizer)**.** *Let $X \to E \to BG$ be a G-action on a space X and let $x \in X$ be a 0-simplex. Then there is an equivalence $G/Stab_x \simeq Orb_x$.*

Proof. Note that by definition we have a fiber sequence

$$Stab_x = \Omega_e X/G \to G \to X \to E \to BG.$$

Therefore we have a fiber sequence $Stab_x \to G \to X$. It follows that, since fibers only depend on the base point component, there is a fiber sequence $Stab_x \to G \to Orb_x$. The proposition then follows from Proposition 6. $\square$

4 Normal Closure

Suppose that $\phi \colon H \to G$ is a map of ∞-groups. We may ask if there is a universal "normal closure" of H through which ϕ factors, i.e. an ∞-group map $\tilde{\phi} \colon \tilde{H} \to G$ such that if $H \to N \to G$ is a factorization of ϕ and $N \to G$ is normal then there is a factorization $H \to \tilde{H} \to N$ over G.

In classical group theory, if $H \to G$ is an injection of groups then the pushout

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G \arrow[d] \\ * \arrow[r] & Z \end{tikzcd}$$

taken in the category of ordinary groups, presents Z as $G/ncl(H)$, where $ncl(H)$ is the smallest normal subgroup of G containing H. We will ask for cofibers in $\mathcal{G}\mathrm{rp}$ to have the same meaning. The challenge is only to extract $ncl(H)$ from $G/ncl(H)$.

Remark 20. Suppose we are given an ∞-group G and map of ∞-groups $\phi\colon G \to K$. Then we can produce a normal map $fib(\phi) \to G$ of ∞-groups (whose associated normality datum is $BG \to BK$). Conversely, given a normality datum $BH \to BG \to Z$ for some $H \to G$, we can produce a map $G \to \Omega Z$. In other words, the normality data for $H \to G$ are precisely the ways in which $H \to G$ can be a fiber of a group map. That motivates the following definition of the *category* of normal maps to G.

Definition 21. Let G be an ∞-group. We define $Sub(G)$ to be the slice category $\mathcal{G}\mathrm{rp}_{/G}$ and $Nml(G)$ to be the coslice category $\mathcal{G}\mathrm{rp}^{\backslash G}$.

Proposition 22. *There is an adjunction between $Sub(G)$ and $Nml(G)$ whose left adjoint $ncl\colon Sub(G) \to Nml(G)$ is given by taking the cofiber (in $\mathcal{G}\mathrm{rp}$) and whose right adjoint is given by taking the fiber (again in $\mathcal{G}\mathrm{rp}$).*

Remark 23. We can reinterpret the above adjunction as being between $(\mathcal{S}_*^{\geq 1})_{/BG}$ and $(\mathcal{S}_*^{\geq 1})^{\backslash BG}$. Given a group map $H \to G$, we deloop it to get $BH \to BG$. We then take the cofiber in $\mathcal{S}_*^{\geq 1}$ (which is the same as taking the cofiber in $\mathcal{G}\mathrm{rp}$ and then delooping) to get

$$\begin{array}{ccc} BH & \longrightarrow & BG \\ \downarrow & & \downarrow \\ * & \longrightarrow & X. \end{array}$$

Now taking the fiber of the map $BG \to X$ gives a map $F \to BG$ and we take $BG \to X$ to be the normality datum for the group map $\Omega F \to G$. Then ΩF is the normal closure of $H \to F$.

Example 24. Taking $G = *$, one computes that the normal closure of an ∞-group $H \to *$ is $\Omega^2 \Sigma BH$. This is the free $\mathbb{E}_2$-space on the $\mathbb{E}_1$-algebra H. Indeed, note the following equivalence of mapping spaces:

$$Map_{\mathbb{E}_2}(\Omega^2 \Sigma BH, Z) \simeq Map_{\mathcal{S}}(\Sigma BH, B^2 Z) \simeq Map_{\mathcal{S}}(BH, BZ) \simeq Map_{\mathbb{E}_1}(H, Z)$$

Proposition 25. *The adjuction of the preceding proposition is monadic. In particular, normality data for a group map $H \to G$ are in bijection with algebra structures on $H \to G$ with respect to the above adjunction.*