Lubin Tate Cohomology and Deformations of n-buds

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Abstract

We reprove Lazard's result that every commutative n-bud is extendible to an $n+1$ bud, from an obstruction theoretic point of view. We locate the obstruction to extending an arbitrary n-bud in a certain cohomology group, and classify isomorphism classes of n-bud extensions for low degrees.

1 Lubin-Tate cohomology

Definition 1. Let R be a commutative, unital ring and F a formal group law on R (associated to a formal group $\mathbb{G}$). We define the Lubin-Tate cosimplicial ring of R with coefficients in $\mathbb{G}$ by the following:

i. $A^n = R[[x_1, \ldots, x_n]]$, where we set $A^0 = R$.

ii. The coface operators $\delta_n^i : A^n \to A^{n+1}$ are defined by $\delta_n^0(f)(x_1, \ldots, x_{n+1}) = f(x_2, \ldots, x_{n+1})$, $\delta_n^{n+1}(f)(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_n)$ and $\delta_n^i(f)(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_i +_F x_{i+1}, \ldots, x_{n+1})$ for i otherwise.

iii. The codegeneracy operators $\sigma_n^i : A^n \to A^{n-1}$ are given by $\sigma_n^i(f)(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, 0, \ldots, x_{n-1})$ for $1 \leq i \leq n$ where the ith entry is replaced by a zero.

It is not hard to check that these maps define a cosimplicial object. The Lubin-Tate complex of R will be the associated alternating sign complex. Let $LT^*(R, \mathbb{G})$ or $LT^*(R, F)$ both denote the graded cohomology ring of this complex, called the Lubin-Tate cohomology of R with coefficients in $\mathbb{G}$ or F respectively (where the use $\mathbb{G}$ or F will depend on context).

Remark 1. Note that the above is just an extension of the second cohomology ring of a formal group law constructed in [11]. We are primarily in this note interested in $LT^*(R, \mathbb{G}_a)$, and primarily in degrees 2 and 3. As such, the only advancement we have past [11] is some investigation of the third degree, and perhaps firmer theoretical footing. Jonathan Lubin suggested that we avoid naming things after mathematicians, so we might want to call it something something like the "formal cohomology" of R, though this is still not a satisfying name.

Remark 2. Also notice that $LT^*(R, \mathbb{G})$ can be realized as the Hochschild cohomology of $\mathbb{G}$ with coefficients in the formal affine R-line (in the sense of II, §3, no. 4.4 of [2]), where the "action" $\mathbb{G} \times \mathbb{A}_R^1 \to \mathbb{A}_R^1$ is the trivial one.

2 n-buds

Definition 2. We define an n-bud over a commutative ring R to be a finite power series $f \in R[[x, y]]$ such that:

i. f is of total degree n or less.

ii. $f(f(x, y), z) - f(x, f(y, z)) \equiv 0$ modulo degree $n + 1$ terms (associativity).

iii. $f(x, 0) = x$ and $f(0, y) = y$ (unitality).

We say that f is also commutative if $f(x, y) = f(y, x)$.

Definition 3. A morphism between an two n-buds f and g in $R[[x, y]]$ is a power series $u(x) \in R[[x]]$ such that $u(f(x, y)) \equiv g(u(x), u(y))$ modulo degree $n + 1$ terms. If the coefficient of x in u is invertible (resp. equal to 1) in R, u is an isomorphism (resp. strict isomorphism).

Notation 1. For brevity, let $Ass(f) = f(f(x, y), z) - f(x, f(y, z))$ and $Ass_n(f)$ be the homogeneous degree n part of $Ass(f)$.

Remark 3. Note that even if the homogeneous degree k part of f is trivial, $Ass_k(f)$ may not be trivial.

Theorem 1 (Lazard)**.** *If $f \in R[[x, y]]$ defines an n-bud, then there exists an $n + 1$-bud $f' \in R[[x, y]]$ such that $f' \equiv f$ modulo degree $n + 1$ if and only if $Ass_{n+1}(f) = \delta_2 h$, for some $h \in R[[x, y]]$. Note that since f is an n-bud, $Ass_k(f) = 0$ for all $k < n + 1$, so h must be of homogeneous degree $n + 1$ (and evidently a finite polynomial). Moreover, $f + h = f'$.*

Proof. Using the Composition Lemma (I.9.10) of [8], we have that $Ass_{n+1}(f + h) = Ass_{n+1}(f) - \delta_2 h$ (this is also formula II.5.7 of [8]). Thus if there is an $n + 1$-bud extending f, it must differ in degree $n + 1$ from f by some finite polynomial h, and if $f + h$ does indeed define an $n + 1$-bud, $Ass_{n+1}(f + h) = 0$. Hence $Ass_{n+1}(f) = \delta_2 h$ if f can be extended. If, on the other hand (and this is the interesting part) $Ass_{n+1}(f) = \delta_2 h$, then $Ass_{n+1}(f + h) = Ass_{n+1}(f) - \delta_2 h = 0$. Note however, we have not shown that for a commutative n-bud with $Ass_{n+1}(f) = \delta_2 h$ for some h that h is symmetric. In other words, we have shown that every commutative n-bud can be extended, but not that every extension is commutative. If R is torsion free or reduced, every bud is commutative, but in general the non-commutative buds (thence non-commutative formal group laws) deserve greater investigation. □

Remark 4. We will later see that $Ass_{n+1}(f)$ is always a coboundary when f is a commutative n-bud. Thus, for such an n-bud, determining which coboundaries $Ass_{n+1}(f)$ can be simultaneously determines all possible extensions of f to an $n + 1$-bud.

Theorem 2. *Let $R[[x]] \ni u(x) : f \to g$ be a morphism of n-buds, and u' a power series such that $u' \equiv u$ modulo degree $n+1$ terms. Let $\Delta_k(r)(f,g)$ denote the homogeneous degree k part of the difference $r(f(x,y)) - r(u(x), u(y))$ for any power series $r \in R[[x]]$. Hence if $u : f \to g$ is a morphism of n-buds, $\Delta_n(u) = 0$. Then $\Delta_{n+1}(u')(f,g) = \Delta_{n+1}(u)(f,g) + \delta_1 h$ for some $h \in R[x]$. In other words, u can be lifted to a morphism of $n+1$-buds if and only if $\Delta_{n+1}(u)(f,g) = \delta_1 h$.*

Proof. Similar to the proof of Theorem 1, and explained in [8]. □

Corollary 1. *Suppose that two $n+1$-buds f' and f'' extend an n-bud f. Then f' and f'' are isomorphic if and only if $f' - f'' = \delta_1 h$ for some $h \in R[x]$.*

Proof. Suppose $f' - f'' = \delta_1 h$. Note that the power series $x \in R[[x]]$ is an automorphism of f. Hence, using the terminology of Theorem 2, we see that $\Delta_{n+1}(x)(f', f'')$ is equal to the degree $n+1$ term of $f' - f''$, which is $\delta_1 h$. Thus, by the preceding theorem, we have a u' which lifts x to a morphism of $n+1$-buds between f' and f''. Moreover, u' is an isomorphism, since it agrees with x below degree $n+1$.

Now, suppose that $\Delta_{n+1}(x)(f', f'') = \delta_1 h$. Then the power series $x + h$ determines a lift of x to an isomorphism between f' and f''.

□

Theorem 3. *For a cochain $f \in R[[x,y]]$, $Ass_n(f)$ is a cocycle in the Lubin-Tate complex.*

Proof. See [8], II, 5.10. □

Corollary 2. *If $LT^3(R, \mathbb{G}_a) = 0$ then every n-bud is extendible to an $n+1$-bud.*

Remark 5. It does not seem to me that this is going to be true very often. However, a weaker result will still imply that every commutative n-bud is extendible. Note that were the antecedent of the preceding corollary satisfied, every n-bud, regardles of commutativity, would be extendible.

Theorem 4. *If $f \in R[x,y,z]$ is a polynomial cocycle such that $f(x,y,z) - f(x,z,y) + f(z,x,y) = 0$, f is the coboundary of a symmetric cochain $h \in R[x,y]$.*

Proof. This is essentially a parameterized version of the same theorem for cocycles in group cohomology given by Eilenberg and Mac Lane in [3], Theorem 26.3. What Eilenberg and Mac Lane show is that the the relative cohomology (with any coefficients) of the pair $(BG, im(\sigma))$ is trivial, where $\sigma : BG \times BG \to BG$ is the shuffle product on cells. In other words, every cocycle is cohomologous to a cocycle which is a coboundary away from shuffle products. This shows that a cocycle $g : G \times G \times G \to G$ is a coboundary if $g(x,y,z) - g(x,z,y) + g(z,x,y) = 0$. The fashion in which they construct this coboundary indicates that it must be the coboundary of a symmetric cochain.

Now, a polynomial cocycle $f \in R[x,y,z]$ determines and is determined by a map $R[x] \to R[x,y,z] \cong R[x] \otimes_R R[y] \otimes_R R[z]$, or dually, a morphism of presheaves $\mathbb{G}_a \times \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ which functorially induces a group cohomology cocycle on the underlying additive group of any R-algebra to which those presheaves are applied. Hence by applying Eilenberg and Mac Lane's argument objectwise (i.e. showing the same statement for the cohomology sheaves of $\mathbb{G}_a$), we obtain the theorem for polynomial cocycles. □

Corollary 3. *Every commutative n-bud is extendible to a commutative $n+1$-bud.*

Proof. Let f define a commutative n-bud over R. Then we have that $Ass_{n+1}(f)$ is a 3-cocycle in the Lubin-Tate complex of R. If f is commutative, it is immediate that $Ass_{n+1}(f)$ satisfies the conditions of Theorem 4. Hence by Theorem 1, we have the corollary. Note that this relies on the commutativity of f. □

3 A Discussion of Commutativity

Some care must be taken in asking about commutative extensions of an arbitrary n-bud (which for the extent of this section will be called f). In his original investigation of n-buds, Lazard considered an n-bud to be a class of power series which agreed modulo degree $n+1$ terms. Here, however, we have taken an n-bud to simply be the power series which is in fact zero above degree n. Note that an extension of f, using our definition, will be commutative if the cochain h determining it is symmetric. However, suppose we used Lazard's definition. We might consider the power series $g = x + y + xy + xy^2 + 2x^2y$ to be a "representative" of our 2-bud $f = x + y + xy$ (suppose $R = \mathbb{Z}$). In this case, $Ass_3(g) = (4-2)xyz = 2xyz$. Thus, using the formula $Ass_3(f) = \delta_2(h)$, we need to find a cochain h such that $\delta_2(h) = 2xyz$. A cochain that satisfies this condition is $h = 2xy^2 + x^2y$. Adding h to f, we get the commutative 3-bud $f' = x + y + xy + 3(xy^2 + x^2y)$, though the cochain by which this extension was obtained was not symmetric. And indeed, the obstruction in this case will not satisfy the conditions of Theorem 4. However, we can remedy the situation with the following:

Proposition 1. *The extensions of an n-bud f determined by a polynomial which is truncated above degree n are in bijection with the extensions of the same n-bud determined by any other power series. Moreover, this bijection preserves commutativity and isomorphism classes of extensions (i.e. it is a groupoid isomorphism!).*

Proof. Let f be a power series $\sum_{i=0}^{\infty} a_{ij}x^iy^j$ determining an n-bud such that $a_{ij} = 0$ whenever $i + j > n$. We will show that symmetric cochains determining extensions of f are in bijection with cochains determining commutative extensions of any other representative of our chosen n-bud. Let g be another representative, let $g|_k$ (resp. $f|_k$) be the homogeneous degree k part of g (resp. f), and let h be a symmetric cochain extending f. Let f' be the commutative extension of f and g determined by adding h to f, and assume that f'

is truncated above degree $n+1$ (this is not integral to the proof but simplifies notation). Then we can deduce that $f' = g + h - g|_{n+1}$ and thus that $Ass_{n+1}(g) = \delta_2(h - g|_{n+1}) = \delta_2(h) - \delta_2(g|_{n+1})$. In other words, there is a set bijection

$$\tau : \left\{ \begin{array}{l} \text{symmetric cochains} \\ \text{extending } f \end{array} \right\} \to \left\{ \begin{array}{l} \text{cochains determining} \\ \text{commutative extensions of } g \end{array} \right\}$$

given by subtraction of $g|_{n+1}$, with inverse being addition of same. It is also clear that this bijection respects isomorphism classes of extensions (in the sense of Theorem 2) and can be extended to non-commutative extensions. □

Hence we can be reassured that determining extensions of an n-bud (in the sense of Lazard) is equivalent to determining extensions of some specific representative thereof. Moreover we now have the following:

Corollary 4. *Isomorphism classes of commutative extensions of an n-bud f are in bijection with homogeneous degree $n+1$ symmetric polynomials $h \in R[x,y]$ such that $\delta_2(h) = Ass_{n+1}(f)$, modulo $\delta_1(R[x])$.*

Theorem 5 (Lazard)**.** *Any two commutative extensions of an n-bud f to $n+1$-buds f' and f'' differ by a symmetric 2-cocycle in $R[x,y]$.*

Proof. [8] □

Corollary 5. *The set of extensions of a commutative n-bud f is a torsor for the cohomology group $LT^2(R, \mathbb{G}_a)$.*

4 Low degree computations

The previous sections allow us to determine an algorithm for computing the moduli of extensions of n-buds.

Example 1. Let

$$f = x + y + a_{11}xy$$

define a 2-bud over a ring R, with $a_{11} \in R$ (note that every 2-bud is commutative). We can compute by hand that $Ass(f) = 0$, so we have that $Ass_3(f) = 0$. In other words, any 2-cocycle $h(x,y) = b_{12}xy^2 + b_{21}x^2y \in R[x,y]$ determines an extension to a three bud $f + h$. So we are interested then in classifying cocycles in 2 variables. Note that

$$\delta_2(h) = (2b_{12} - 2b_{21})xyz,$$

so we're attempting to solve the equation $2b_{12} - 2b_{21} = 0$ over R. Thus, if R has no 2-torsion, every 3 bud must be commutative. However, if R has 2-torsion, then choosing b_{12} and b_{21} to be 2-torsion elements will also produce a cocycle which need not be symmetric. So we see that commutative extensions are in bijection with R, but arbitrary extensions involve choosing an element of the torsion free part of R (possibly zero!) and two elements of the 2-torsion of R. Note that there is a group structure on extensions (just add coefficients) since δ_2 is additive.

We also note here that the above analysis implies that every single commutative extension of the multiplicative 2-bud (also an honest group scheme!) to a commutative $k > 2$-bud is determined by the choice of a homogeneous degree k symmetric 2-cocycle. And since isomorphic extensions differ by coboundaries, the extensions of the $x + y + cxy$ are just the cohomology classes of $LT^2(R, \mathbb{G}_a)$.

Example 2. Now suppose we have a 3-bud

$$f(x, y) = x + y + a_{11}xy + a_{12}xy^2 + a_{21}x^2y$$

and we want to determine ways that it can be extended to a 4-bud. We calculate

$$Ass_4(f) = (2a_{11}a_{21})x^2yz + (a_{11}a_{21} - a_{11}a_{12})xy^2z + (2a_{11}a_{12})xyz^2$$

Thus we must find $h \in R[x, y]$ such that $\delta_2 h = Ass_4(f)$. Now, an arbitrary normalized 2-cochain of homogeneous degree 4 is of the form

$$h(x, y) = b_{13}xy^3 + b_{22}x^2y^2 + b_{31}x^3y$$

and has coboundary

$$\delta_2(h) = (-3b_{31} + 2b_{22})x^2yz + (3b_{13} - 3b_{31})xy^2z + (3b_{13} - 2b_{22})xyz^2.$$

Thus to determine possible extensions we see that we have to solve the system of equations:

$$\begin{aligned} 3b_{31} - 2b_{22} &= 2a_{11}a_{21} \\ 3b_{31} - 3b_{13} &= a_{11}a_{21} - a_{11}a_{12} \\ 2b_{22} - 3b_{13} &= -2a_{11}a_{12}. \end{aligned}$$

It should be clear that if the ring R has no 2-torsion or 3-torsion elements then a choice of any one of b_{22}, b_{13} or b_{31} determines the other two. Notice that we have not yet said anything about f or h being commutative. An interesting example is the case in which R has 2-torsion and we are interested in extending a non-commutative 3-bud, for instance $f(x) = x + y + xy + cxy^2$, where c is a 2-torsion element of R. Then we see that we must simultaneously have $b_{13} = b_{31} = 0$ and $0 = 3b_{31} - 3b_{13} = -c$, hence no such extension can exist.

There are obviously many cases to be considered further, e.g. if we assume that f is commutative, or if we assume that R is 3-torsion. In the case that f is non-commutative and R is 3-torsion, it's clear that rather strong conditions must be put on f to extend it to a 4-bud.

Example 3. Lastly we consider extending a 4-bud

$$f(x, y) = x + y + a_{11}xy + a_{12}xy^2 + a_{21}x^2y + a_{13}xy^3 + a_{22}x^2y^2 + a_{31}x^3y$$

to a 5-bud. We first compute:

$$\begin{aligned} Ass_5(f) = &(2a_{21}^2 + 3a_{11}a_{31})x^3yz \\ &+ (a_{11}^2a_{21} + a_{21}^2 + 2a_{21}a_{12} - a_{11}a_{22} + 6a_{11}a_{31})x^2y^2z \\ &+ (2a_{11}a_{31} - 2a_{11}a_{13}+)xy^3z \\ &+ (-a_{11}^2a_{12} - a_{12}^2 - 2a_{21}a_{12} + a_{11}a_{22} - 6a_{11}a_{13})xy^2z^2 \\ &+ (-2a_{12}^2 - 3a_{11}a_{13})xyz^3 \end{aligned}$$

and

$$\begin{aligned} \delta_2(h) = &(2b_{32} - 4b_{41})x^3yz \\ &+ (-6b_{41} + 3b_{23})x^2y^2z \\ &+ (4b_{14} - 4b_{41})xy^3z \\ &+ (-3b_{32} + 3b_{23})x^2yz^2 \\ &+ (6b_{14} - 3b_{32})xy^2z^2 \\ &+ (4b_{14} - 2b_{23})xyz^3 \end{aligned}$$

So we get a system of equations we need to solve:

$$\begin{aligned} &4b_{41} - 2b_{32} = 2a_{21}^2 + 3a_{11}a_{31} \\ &6b_{41} - 3b_{23} = a_{11}^2a_{21} + a_{21}^2 + 2a_{21}a_{12} + 6a_{11}a_{31} - a_{11}a_{22} \\ &4b_{41} - 4b_{14} = 2a_{11}a_{31} - 2a_{11}a_{13} \\ &3b_{32} - 3b_{23} = 0 \\ &3b_{32} - 6b_{14} = -a_{11}^2a_{12} - a_{12}^2 - 2a_{21}a_{12} - 6a_{11}a_{13} + a_{11}a_{22} \\ &2b_{23} - 4b_{14} = -2a_{12}^2 - 3a_{11}a_{13} \end{aligned}$$

Again, if we're working over a torsion free ring, a choice of a single b_{ij} determines the rest.

5 Computations in the non-smooth case

Now we might be interested repeating the above process for a slightly more interesting affine scheme, for instance let $A = R[x, y]/(x^3 - y^2)$ so that $X = Spec(A)$ is a curve with a cusp at the origin. As such, one might hope that there are non-trivial obstructions in this case (even assuming commutativity). First we recall some definitions:

Definition 4. A 2-dimensional formal group law $R[[x, y]] \to R[[x_1, x_2, y_1, y_2]]$ is a pair of 4 variable power series $F_1(x_1, x_2, y_1, y_2)$ and $F_2(x_1, x_2, y_1, y_2)$ satifying:

i. $F_1(x_1,x_2,y_1,y_2) \equiv x_1+y_1$ and $F_2(x_1,x_2,y_1,y_2) \equiv x_2+y_2$ modulo terms of degree greater than 1.

ii. $F_1(F_1(x_1,x_2,y_1,y_2),F_2(x_1,x_2,y_1,y_2),z_1,z_2) = F_1(x_1,x_2,F(y_1,y_2,z_1,z_2),G(y_1,y_2,z_1,z_2))$, and $F_2(F_1(x_1,x_2,y_1,y_2),F_2(x_1,x_2,y_1,y_2),z_1,z_2) = F_2(x_1,x_2,F_1(y_1,y_2,z_1,z_2),F_2(y_1,y_2,z_1,z_2))$.

Definition 5. Suppose (F_1,F_2) defines a 2-dimensional n-bud. Let $Ass^1_{n+1}(F_1,F_2)$ be the degree $n+1$ part of the difference

$$F_1(F_1(x_1,x_2,y_1,y_2),F_2(x_1,x_2,y_1,y_2),z_1,z_2) - F_1(x_1,x_2,F(y_1,y_2,z_1,z_2),G(y_1,y_2,z_1,z_2))$$

and let $Ass^2_{n+1}(F_1,F_2)$ be the degree $n+1$ part of the difference

$$F_2(F_1(x_1,x_2,y_1,y_2),F_2(x_1,x_2,y_1,y_2),z_1,z_2) - F_2(x_1,x_2,F_1(y_1,y_2,z_1,z_2),F_2(y_1,y_2,z_1,z_2)).$$

Lemma 1. *Suppose (F_1,F_2) is a 2-dimensional n-bud. And let (F_1+h_1,F_2+h_2) extend (F_1,F_2). Then we have that*

$$Ass^1_{n+1}(F_1+h_1,F_2+h_2) = Ass^1_{n+1}(F_1,F_2) - \delta' h_1,$$

and

$$Ass^2_{n+1}(F_1+h_1,F_2+h_2) = Ass^2_{n+1}(F_1,F_2) - \delta' h_2,$$

where for $h \in R[x_1,x_2,y_1,y_2]$,

$$\begin{aligned}\delta' h(x_1,x_2,y_1,y_2,z_1,z_2) = (h(y_1,y_2,z_1,z_2) &- h(x_1+y_1,x_2+y_2,z_1,z_2) \\ &+ h(x_1,x_2,y_1+z_1,y_2+z_2) - h(x_1,x_2,y_1,y_2).\end{aligned}$$

Proof. This again follows directly from the Composition Lemma in [8]. □

Corollary 6. *Identically to the discussion in the preceding sections, we know that $Ass^1_{n+1}(F_1,F_2)$ and $Ass^2_{n+1}(F_1,F_2)$ are the obstructions to extending a given 2-dimensional n-bud (F_1,F_2).*

Thus we'd like to define a cohomology theory in which these obstructions live. Indeed, it's basically the cohomology theory given in the first section with an extra pair of variables. In other words, we have the cochain complex

$$R[[x,y]] \to R[[x,y,z,w]] \to R[[x,y,z,w,s,t]] \to \dots$$

where the first coboundary is $f(x,y) \mapsto f(z,w) - f(x+z,y+w) + f(x,y)$ and the second is $f(x,y,z,w) \mapsto f(z,w,s,t) - f(x+z,y+w,s,t) + f(x,y,z+s,w+t) - f(x,y,z,w)$, and so forth in higher degrees.

Thus, the obstruction to extending a 2-dimensional n-bud clearly lives in this complex, and the story goes exactly the same.

The following example was pointed out to me by Paul VanKoughnett.

Example 4 (Cuspidal $\hat{\mathbb{A}}^2$)**.** The additive formal group law in two dimensions is given by the pair of of power series:

$$F_1(x_1, x_2, y_1, y_2) = x_1 + y_1 \text{ and } F_2(x_1, x_2, y_1, y_2) = x_2 + y_2$$

It's not hard to check that is satisfies the necessary conditions. Moreover, it's commutative. However, suppose we wish to have the additive formal group define a cogroup structure at a cusp:

$$R[[x_1, x_2]]/(x_1^2 - x_2^3) \to R[[x_1, x_2]]/(x_1^2 - x_2^3)\hat{\otimes}R[[y_1, y_2]]/(y_1^2 - y_2^3)$$

Then we are immediately confronted by the issue that the additive group is not even well defined. For instance, our group law should take "zero" to "zero." Let $f(x_1, x_2) = x_1^2 - x_2^3 \equiv 0$.

Then

$$\begin{aligned} f(x_1 + y_1, y_2 + y_2) &= (x_1 + y_1)^2 - (x_2 + y_2)^3 \\ &= x_1^2 + 2x_1y_1 + y_1^2 - x_2^3 - 3x_2^2y_2 - 3x_2y_2^2 - y_2^3 \\ &\equiv 2x_1y_1 - 3(x_2^2y_2 + x_2y_2^2). \end{aligned}$$

The final equivalence above is given by reducing modulo the relations $x_1^2 = x_2^3$ and $y_1^2 = y_2^3$. If we reduce modulo terms of degree greater than 1, we see that the additive group defines a cogroup structure (i.e. a group structure on the tangent plane of the given cusp at the origin), but we cannot necessarily extend it to higher degree terms. Note that if we are interested in 2-buds, we have to solve the equation $0 = 2x_1y_1$. Hence, if our ring is all 2-torsion, the additive group defines what one might call a "reduced 2-bud" on the given cusp (i.e. a group structure on the second degree approximation of the given cusp at the origin).

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