Matroids as $\mathbb{F}_1$-modules

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1 Formal Projective Geometry

In 1910, Oswald Veblen introduced an axiomatic approach to (projective) geometry [Veb07, VY08] thought of as the data of two sets $(G, L \subseteq \mathcal{P}(G))$ satisfying certain properties that made the set L behave like the set of *lines* in a the set G of *points*.

In 1943, Walter Prenowitz reinterpreted Veblen's (and Young's) approach in terms of what he called a *multigroup*. This is a set G along with a function $\ell\colon G \times G \to \mathcal{P}(G)$ satisfying certain conditions. We are supposed to think of $\ell(x, y)$ as the *unique line containing x and y*. Prenowitz showed that all projective geometries in the sense of Veblen gave multigroups but that multigroups were slightly more general. In Veblen's geometries a line was required to contain *at least* three points (the idea being of course that in any kind of reasonable projective geometry one shouldn't have a "line" whose only points are 0 and ∞). The "projective geometries" associated to Prenowitz's multigroups only required their lines to have at least *two* points. With this adjustment, the two notions are entirely equivalent.

Meanwhile, in the 1930's, both Hassler Whitney and Takeo Nakasawa were introducing the ideas that would come to be referred to as *matroids* as a common generalization of both linear algebra/projective geometry and graph theory [Whi35, NK09]. There are many equivalent definitions of matroids (cf. "cryptomorphisms") but the one which is the most efficient for this talk is the following:

Definition 1. A *simple pointed matroid* is a set Z equipped with a function $C\colon \mathcal{P}(Z) \to \mathcal{P}(Z)$, called a *closure operator* and a distinguished element $0 \in Z$ satisfying the following conditions:

1. For all $X \subseteq Z$, $X \subseteq C(X)$.
2. For all $Y \subseteq X \subseteq Z$, $C(Y) \subseteq C(X)$.
3. For all $X \subseteq Z$, $C(C(X)) = C(X)$.
4. For all $x, y \in Z$ and $X \subseteq Z$, if $x \notin C(X)$ and $x \in C(X \cup \{y\})$ then $y \in C(X \cup \{y\})$.
5. $C(\varnothing) = \{0\}$ and $C(\{x\}) = \{x, 0\}$.

The category whose objects are simple pointed matroids and whose morphisms are functions that preserve base points and commute with closure operators is denoted $sMat_*$.

Inside of $sMat_*$ is a full subcategory of objects called *projective geometries* in [NR23] (and they are, indeed, equivalent to the classical axiomatic characterizations of projective geometries). We'll write this full subcategory as $Proj$.

Perhaps following in Prenowitz's footsteps, we might now ask if these versions of projective geometries can be realized by some kind of hyperoperation. For this we need a slightly more general notion than Prenowitz's multigroups:

Definition 2. A *weakly unital commutative hypermagma* is a set M along with a distinguished element $e \in M$ and a function $\star\colon M \times M \to \mathcal{P}(M)$ satisfying the following properties:

1. For all $x, y \in M$, $x \star y = y \star x$.
2. For all $x \in M$, $x \in (x \star e) \cap (e \star x)$.

There is a category of weakly unital commutative hypermagmas with morphisms functions that preserve base points and such that $f(x \star y) \subseteq f(x) \star f(y)$. Denote this category $CHMag^u$.

Proposition 3. [NR23] *Let $(Z, C, 0)$ be a simple pointed matroid. Then there is a weakly unital commutative hypermagma $\bar{Z} = (Z, \star, 0)$ with operation given by*

$$x \star y = \begin{cases} C(\{x, y\}) & x \neq y \\ \{x, 0\} & x = y \end{cases}$$

Nakamura and Reyes prove the following:

Theorem 4. [NR23] *The construction $Z \mapsto \bar{Z}$ extends to a faithful functor $sMat_* \to CHMag^u$. When restricted to $Proj \subseteq sMat_*$, this functor is fully faithful.*

So, similarly to Prenowitz, we can think of $CHMag^u$ as being a (vast) generalization of projective geometries which happens to include (simple, pointed) matroids, which are independently interesting objects because of their connections to graph theory, combinatorics, number theory, and other areas of mathematics.

2 Hyperoperations, Γ-sets and $\mathbb{F}_1$

Around the same time as Prenowitz, Jacques Tits [Tit11] was noticing that certain patterns and formulae in projective geometry (in the usual sense) over finite fields seemed to have alarmingly similar analogues in the combinatorics of *finite sets*. These analogues seemed to, in many cases, be obtained by taking the copy of q that appeared in formulae for projective geometry over $\mathbb{F}_q$, and replacing it with 1. He hypothesized that combinatorics could be *thought of* as something like *projective geometry over a field of characteristic one*. Of course this doesn't make sense on the face of it, but the general idea is still compelling. The basic dictionary looks something like

$$\begin{aligned} \mathbb{F}_q &\mapsto \text{“}\mathbb{F}_1\text{”} \\ \mathbb{F}_q^n &\mapsto \{1, 2, 3, \dots, n\} \\ GL_n(\mathbb{F}_q) &\mapsto \Sigma_n \end{aligned}$$

Of course the second line above suggests that $\mathbb{F}_1$ should just be a one element set, but I wrote $\mathbb{F}_1$ in scare quotes to indicate that it will necessarily need to have *some* structure besides being a singleton set, if we're going to get anywhere with it.

Later on, this idea gained even more traction when Kapranov, Smirnov and Manin [Smi92, Man95] suggested that if there *were* a field of characteristic one, call it $\mathbb{F}_1$, then $Spec(\mathbb{Z})$ would be a curve over $Spec(\mathbb{F}_1)$ and so Deligne's proof of the Weil conjectures *for function fields* might be replicable in such a setting (thereby proving the Riemann Hypothesis).

To actually *use* $\mathbb{F}_1$ for any of this however, one must actually have an actual mathematical object *called* $\mathbb{F}_1$ to study, or at the very least a category of modules or vector spaces *over it*. There are *many* models for $\mathbb{F}_1$ that have been floated since Tits' seminal work, but for this talk I'm interested in the one used by Connes and Consani (in their quest to prove the Riemann Hypothesis) [CC16].

Definition 5. Let Fin_* be the skeleton of the category of finite pointed sets and pointed maps spanned by the objects $\langle n \rangle = \{0, 1, 2, \dots, n\}$, where 0 is taken to be the base point. Let Set_* be the category of all finite pointed sets and pointed maps between them. Then we define the category of $\mathbb{F}_1$-modules to be

$$\mathrm{Mod}_{\mathbb{F}_1} = \mathrm{Fun}_*(\mathrm{Fin}_*, \mathrm{Set}_*)$$

where the notation Fun_* denotes functors which take $\langle 0 \rangle$ to a singleton set.

We take $\mathbb{F}_1$ itself to be the inclusion functor $\mathrm{Fin}_* \hookrightarrow \mathrm{Set}_*$. This category is equipped with the *Day convolution* monoidal structure and, indeed, $\mathbb{F}_1$ is the monoidal unit therein, so this category is a category of modules over the object we've called $\mathbb{F}_1$.

Remark 6. Note that in the above definition $\mathbb{F}_1$ is the functor corepresented by the pointed set $\{0, 1\}$. This is consistent with a number of past approaches to $\mathbb{F}_1$ which identify it as the set $\{0, 1\}$ with some additional structure.

In [CC16], it is shown that this category contains, or at least admits functors from, lots of interesting structures. For instance, the generalized rings of Durov (a.k.a. Lawvere theories) give examples [Dur07], as well as commutative monoids, and finite sets themselves (via the coYoneda embedding). There is also a way to think of certain combinatorial structures, encoded as commutative monoids in the span category $Span(\mathrm{Set}_*)$, as $\mathbb{F}_1$-modules [CMS23]. Being symmetric monoidal, $\mathrm{Mod}_{\mathbb{F}_1}$ admits a theory of *algebras* and the inclusion from commutative monoids is lax monoidal, therefore it takes (semi)rings to $\mathbb{F}_1$-algebras. This allows one to think of certain structures from tropical geometry and Arakelov geometry as $\mathbb{F}_1$-algebras as well.

Of particular interest to us is the way in which one can encode *hyper*structures in $\mathrm{Mod}_{\mathbb{F}_1}$. This is important to Connes and Consani because they want to understand their *adele class space* $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$, (which is naturally a hyperring) as an $\mathbb{F}_1$-algebra.

To understand functors $\mathrm{Fin}_* \to \mathrm{Set}_*$, it will be helpful to have some names for some of the maps in Fin_*.

Definition 7. Write $\langle n \rangle = \{*, 1, 2, \dots, n\}$ for the objects of Fin_*. Then write:

1. $\rho_i \colon \langle n \rangle \to \langle 1 \rangle$ for the map given by
$$\rho_i(j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

2. $\mu \colon \langle 2 \rangle \to \langle 1 \rangle$ for the map given by $\mu(1) = \mu(2) = 1$.

3. $e \colon \langle 0 \rangle \to \langle 1 \rangle$ for the uniqe pointed map.

Definition 8. Let $X \colon \mathrm{Fin}_* \to \mathrm{Set}_*$. Write
$$S_n \colon X\langle n \rangle \to \prod_n X\langle 1 \rangle$$
for the map induced by the maps $X(\rho_i) \colon X\langle n \rangle \to X\langle 1 \rangle$ defined above.

Definition 9. Let A be a commutative monoid. We (roughly) define a functor $HA \colon \mathrm{Fin}_* \to \mathrm{Set}_*$ by setting $HA\langle n \rangle = A^n$, defining $HA(\rho_i)$ to be the various projection maps from the product, $HA(\mu)$ to be the addition map of A (as well as higher copies of μ obtained by taking coproducts of it with identities), HA of permutations to be permutations of the coordinates, and $HA(e)$ is the unit map $\{*\} \to A$.

Proposition 10. *Let X be a functor $\mathrm{Fin}_* \to \mathrm{Set}_*$. Then the maps S_n are isomorphisms if and only if $X \cong HA$ for some commutative monoid A. Moreover, there is a fully faithful functor $H \colon CMon \to \mathrm{Mod}_{\mathbb{F}_1}$.*

The condition of the S_n maps being isomorphisms is often called the "Segal condition." But Connes and Consani do not require that their objects satisfy the Segal condition. They consider arbitrary functors. The reason for this becomes clear if we consider the following diagram obtained by applying a functor X to Fin_*:

$$\begin{array}{ccc} X\langle 2 \rangle & \xrightarrow[S_2]{} & X\langle 1 \rangle \times X\langle 1 \rangle \\ {\scriptstyle X\mu}\big\downarrow & & \\ X\langle 1 \rangle & & \\ {\scriptstyle Xe}\big\uparrow & & \\ X\langle 0 \rangle & & \end{array}$$

If S_2 is an isomorphism then we can take its inverse to obtain the "addition" $X\langle 1 \rangle \times X\langle 1 \rangle \cong X\langle 2 \rangle \xrightarrow{X\mu} X\langle 1 \rangle$, and one checks that Xe is a unit for this operation. In the case that S_2 is not an isomorphism, we can still take its "inverse," it's just that the inverse takes pairs (x, y) to *subsets* of $X\langle 2 \rangle$. Therefore the composite $X\mu \circ S_2^{-1}$ is a hyperoperation $X\langle 1 \rangle \times X\langle 1 \rangle \to \mathcal{P}(X\langle 1 \rangle)$. By a relatively straightforward diagram chase, one checks that this defines a functor:

Proposition 11 (B.-Nakamura)**.** *There is a functor $T \colon \mathrm{Mod}_{\mathbb{F}_1} \to CHMag^u$ which takes X to the set $X\langle 1 \rangle$ with the hyperoperation $X\mu \circ S_2^{-1}$ and weak unit Xe.*

More importantly, we can produce $\mathbb{F}_1$-modules from weakly unital commutative hypermagmas.

Theorem 12 (B.-Nakamura)**.** *The functor T has a fully faithful right adjoint $\hat{H}\colon CHMag^u \to \mathrm{Mod}_{\mathbb{F}_1}$ which, when restricted to $CMon$, is naturally isomorphic to the "Eilenberg-MacLane" functor H.*

Corollary 13. *The composite of $\hat{H}$ with the Nakamura and Reyes' functor on simple pointed matroids gives a faithful inclusion $sMat_* \to \mathrm{Mod}_{\mathbb{F}_1}$ and a fully faithful inclusion $Proj \hookrightarrow \mathrm{Mod}_{\mathbb{F}_1}$*

The functor is somewhat complicated, so I won't define it here, but at low levels it's easy to define. If $(M, \star, e)$ is a weakly unital commutative hypermagma then on the first objects we get

$$\hat{H}M\langle 0\rangle = \{*\}$$
$$\hat{H}M\langle 1\rangle = M$$
$$\hat{H}M\langle 2\rangle = \{(a,b,c) \in M^3 \colon b \in a \star c\}$$

For morphisms, $\hat{H}M(\rho_1), \hat{H}M(\rho_2)\colon \hat{H}M\langle 2\rangle \to \hat{H}M\langle 1\rangle$ are given by projection onto the left and right coordinates and $\mu\colon \hat{H}M\langle 2\rangle \to \hat{H}M\langle 1\rangle$ is given by projection on the center coordinate. Of course $\hat{H}M(e)$ is the morphism that picks out $e \in M$. It's not difficult to see that $T\hat{H}M = M$.

Remark 14. The functor T is very close to the pullback functor $\mathrm{Fun}_*(\mathrm{Fin}_*, \mathrm{Set}_*) \to \mathrm{Fun}_*(\mathrm{Fin}_*^{\leq 2}, \mathrm{Set}_*)$ where $\mathrm{Fin}_*^{\leq 2}$ denotes the full subcategory of Fin_* spanned by the objects $\langle 0\rangle$, $\langle 1\rangle$ and $\langle 2\rangle$. This is not surprising, as the information that T isolates is entirely contained in $\mathrm{Fin}_*^{\leq 2}$. One can then show that the functor $\hat{H}$ is essentially the same as right Kan extending an object of $\mathrm{Fun}_*(\mathrm{Fin}_*^{\leq 2}, \mathrm{Set}_*)$ along the inclusion $\mathrm{Fin}_*^{\leq 2} \hookrightarrow \mathrm{Fin}_*$.

References

[CC16] Alain Connes and Caterina Consani, *Absolute algebra and Segal's Γ-rings: au dessous de* $\overline{\mathrm{Spec}(\mathbb{Z})}$, J. Number Theory **162** (2016), 518–551.

[CMS23] Ivan Contreras, Rajan Amit Mehta, and Walker H. Stern, *Frobenius and commutative pseudomonoids in the bicategory of spans*, 2023. arXiv:2311.15342.

[Dur07] Nikolai Durov, *New approach to arakelov geometry*, 2007. arXiv:0704.2030.

[Man95] Yuri Manin, *Lectures on zeta functions and motives (according to Deninger and Kurokawa)*, 1995, pp. 4, 121–163. Columbia University Number Theory Seminar (New York, 1992). MR1330931

[NK09] Hirokazu Nishimura and Susumu Kuroda (eds.), *A lost mathematician, Takeo Nakasawa. The forgotten father of matroid theory*, Basel: Birkhäuser, 2009 (English).

[NR23] So Nakamura and Manuel L. Reyes, *Categories of hypermagmas, hypergroups, and related hyperstructures*, 2023. arXiv: 2304.09273.

[Smi92] A. L. Smirnov, *Hurwitz inequalities for number fields*, Algebra i Analiz **4** (1992), no. 2, 186–209. MR1182400

[Tit11] J. Tits, *Sur les groupes algébriques affins. Théorèmes fondamentaux de structure. Classification des groupes semisimples et géométries associées*, Groups, Lie rings and cohomology theory (Italian), 2011, pp. 185–264. MR2885226

[Veb07] Oswald Veblen, *Collineations in a finite projective geometry*, Trans. Amer. Math. Soc. **8** (1907), no. 3, 366–368. MR1500790

[VY08] Oswald Veblen and John Wesley Young, *A Set of Assumptions for Projective Geometry*, Amer. J. Math. **30** (1908), no. 4, 347–380. MR1506049

[Whi35] Hassler Whitney, *On the Abstract Properties of Linear Dependence*, Amer. J. Math. **57** (1935), no. 3, 509–533. MR1507091